SUFFICIENT AND NECESSARY CONDITIONS FOR SEMIDEFINITE REPRESENTABILITY OF CONVEX HULLS AND SETS

J. WILLIAM HELTON AND JIAWANG NIE

ABSTRACT. A set $S \subseteq \mathbb{R}^n$ is called to be *Semidefinite (SDP)* representable if S equals the projection of a set in higher dimensional space which is describable by some Linear Matrix Inequality (LMI). Clearly, if S is SDP representable, then S must be convex and semialgebraic (it is describable by conjunctions and disjunctions of polynomial equalities or inequalities). This paper proves sufficient conditions and necessary conditions for SDP representability of convex sets and convex hulls by proposing a new approach to construct SDP representations.

The contributions of this paper are: **(i)** For bounded SDP representable sets $W_1, \cdots, W_m$, we give an explicit construction of an SDP representation for $\mathrm{conv}(\cup_{k=1}^m W_k)$. This provides a technique for building global SDP representations from the local ones. **(ii)** For the SDP representability of a compact convex semialgebraic set S, we prove sufficient: the boundary ∂S is nonsingular and positively curved, while necessary is: ∂S has nonnegative curvature at each nonsingular point. In terms of defining polynomials for S, nonsingular boundary amounts to them having nonvanishing gradient at each point on ∂S and the curvature condition can be expressed as their strict versus nonstrict quasi-concavity of at those points on ∂S where they vanish. The gaps between them are ∂S having or not having singular points either of the gradient or of the curvature's positivity. A sufficient condition bypassing the gaps is when some defining polynomials of S satisfy an algebraic condition called sos-concavity. **(iii)** For the SDP representability of the convex hull of a compact nonconvex semialgebraic set T, we find that the critical object is $\partial_c T$, the maximum subset of ∂T contained in $\partial \mathrm{conv}(T)$. We prove sufficient for SDP representability: $\partial_c T$ is nonsingular and positively curved, and necessary is: $\partial_c T$ has nonnegative curvature at nonsingular points. The gaps between our sufficient and necessary conditions are similar to case (ii). The positive definite Lagrange Hessian (PDLH) condition, which meshes well with constructions, is also discussed.

1. INTRODUCTION

Semidefinite programming (SDP) [1, 9, 10, 14] is one of the main advances in convex optimization theory and applications. It has a profound effect on combinatorial optimization, control theory and nonconvex optimization as well as many other disciplines. There are effective numerical algorithms for solving problems presented in terms of Linear Matrix Inequalities (LMIs). One fundamental problem in semidefinite programming and linear matrix inequality theory is what sets can be presented in semidefinite programming. This paper addresses one of the most classical aspects of this problem.

A set S is said to have an *LMI representation* or be *LMI representable* if

$$S = \{x \in \mathbb{R}^n : A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}$$

for some symmetric matrices A_i. Here the notation $X \succeq 0 \, (\succ 0)$ means the matrix X is positive semidefinite (definite). If S has an interior point, A_0 can be assumed to be positive definite without loss of generality. Obvious necessary conditions for S to be LMI representable are that S must be convex and *basic closed semialgebraic*, i.e.,

$$S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$$

where $g_i(x)$ are multivariate polynomials. It is known that not every convex basic closed semialgebraic set can be represented by LMI (e.g., the set $\{x \in \mathbb{R}^2 : x_1^4 + x_2^4 \leq 1\}$ is not LMI representable [7]). If the

---

convex set S can be represented as the projection to $\mathbb{R}^n$ of

$$(1.1) \qquad \hat{S} = \left\{ (x,u) \in \mathbb{R}^{(n+N)} : A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{N} B_j u_j \succeq 0 \right\} \subset \mathbb{R}^{(n+N)},$$

that is $S = \left\{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^n, \ (x,u) \in \hat{S} \right\}$, for some symmetric matrices A_i and B_j, then S is called *semidefinite representable* or *SDP representable*. Sometimes we refer to a semidefinite representation as a *lifted LMI representation* of the convex set S and to the LMI in (1.1) as a *lifted LMI for S*, and to $\hat{S}$ as the *SDP lift of S*.

If S has an SDP representation instead of LMI representation, then S might not be basic closed semialgebraic, but it must be semialgebraic, i.e., S is describable by conjunctions or disjunctions of polynomial equalities or inequalities [3]. Furthermore, the interior $\overset{\circ}{S}$ of S is a union of *basic open semialgebraic sets* (Theorem 2.7.2 in [3]), i.e., $\overset{\circ}{S} = \bigcup_{k=1}^{m} T_k$ for sets of the form

$$T_k = \{ x \in \mathbb{R}^n : g_{j_1}(x) > 0, \cdots, g_{j_{m_k}}(x) > 0 \}.$$

Here g_{i_j} are all multivariate polynomials. For instance, the set

$$\left\{ x \in \mathbb{R}^2 : \exists u \geq 0, \begin{bmatrix} x_2 & x_1 - u \\ x_1 - u & 1 \end{bmatrix} \succeq 0 \right\}$$

is not a basic semialgebraic set. When S is SDP representable, S might not be closed, but its closure $\bar{S}$ is a union of basic closed semialgebraic sets (Proposition 2.2.2 and Theorem 2.7.2 in [3]). For example, the set

$$\left\{ x \in \mathbb{R} : \exists u, \begin{bmatrix} x & 1 \\ 1 & u \end{bmatrix} \succeq 0 \right\} = \{ x \in \mathbb{R} : x > 0 \}$$

is not closed, but its closure is a basic closed semialgebraic set. The content of this paper is to give sufficient conditions and (nearby) necessary conditions for SDP representability of convex semialgebraic sets or convex hulls of nonconvex semialgebraic sets.

History   Nesterov and Nemirovski ([9]), Ben-Tal and Nemirovski ([1]), and Nemirovsky ([10]) gave collections of examples of SDP representable sets. Thereby leading to the fundamental question which sets are SDP representable? In §4.3.1 of his excellent ICM 2006 survey [10] Nemirovsky wrote " this question seems to be completely open". Obviously, to be SDP representable, S must be convex and semialgebraic. What are the sufficient conditions that S is SDP representable? This is the main subject of this paper.

When S is a basic closed semialgebraic set of the form $\{ x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0 \}$, there is recent work on the SDP representability of S and its convex hull. Parrilo [11] and Lasserre [8] independently proposed a natural construction of lifted LMIs using moments and sum of squares techniques with the aim of producing SDP representations. Parrilo [11] proved the construction gives an SDP representation in the two dimensional case when the boundary of S is a single rational planar curve of genus zero. Lasserre [8] showed the construction can give arbitrarily accurate approximations to compact S, and the constructed LMI is a lifted LMI for S by assuming *almost all* positive affine functions on S have SOS representations with uniformly bounded degree. Helton and Nie [6] proved that this type of construction for compact convex sets S gives the exact SDP representation under various hypotheses on the Hessians of the defining polynomials $g_i(x)$, and also gave other sufficient conditions for S to be SDP representable. Precise statements of most of the main theorems in [6] can be seen here in this paper in later sections where they are used in our proofs, see Theorems 3.1, 5.2 and 5.3.

Contributions   In this paper, we prove sufficient and (nearby) necessary conditions for the SDP representability of convex sets and convex hulls of nonconvex sets. To obtain these conditions we give a new and different construction of SDP representations, which we combine with those discussed in [6, 8, 11]. The following are our main contributions.

First, consider the SDP representability of the convex hull of union of sets $W_1, \cdots, W_m$ which are all SDP representable. When every W_k is bounded, we give an explicit SDP representation of $\text{conv}(\cup_{k=1}^m W_k)$. When some W_k is unbounded, we show that the closure of the projection of the constructed SDP lift is exactly the closure of $\text{conv}(\cup_{k=1}^m W_k)$. This is Theorem 2.2. It provides a new approach for constructing global SDP representations from local ones, and plays a key role in proving our main theorems in Sections §3 and §4.

Second, consider the SDP representability of a compact convex semialgebraic set $S = \cup_{k=1}^m T_k$. Here $T_k = \{x \in \mathbb{R}^n : g_1^k(x) \geq 0, \cdots, g_m^k(x) \geq 0\}$ are defined by polynomials g_i^k; note each T_k here is not necessarily convex. Denote by $Z(g)$ the zero set of a polynomial g. Our main result for everywhere nonsingular boundary ∂S is approximately:

Assume each T_k has interior near $\partial T_k \cap \partial S$ and its boundary is nonsingular (the defining polynomials g_i^k at every point $u \in \partial S \cap Z(g_i^k)$ satisfy $\nabla g_i^k(u) \neq 0$). Then sufficient for S to be SDP representable is: every $\partial S \cap Z(g_i^k)$ has positive curvature (i.e. g_i^k is strictly quasi-concave on $\partial S \cap Z(g_i^k)$), and necessary is: every irredundant $Z(g_i^k)$ has nonnegative curvature on ∂S (i.e. g_i^k is quasi-concave at u whenever $Z(g_i^k)$ is irredundant at $u \in \partial S \cap Z(g_i^k)$).

The notion of positive curvature we use is the standard one of differential geometry, the notion of quasi-concave function is the usual one and all of this will be defined formally in §3. To have necessary conditions on a family F of defining functions for S we need an assumption that F contains no functions irrelevant to the defining of S. Our notion of irredundancy plays a refinement of this role. The gaps between our sufficient and necessary conditions are ∂S having positive versus nonnegative curvature and singular versus nonsingular points. A case bypassing the gaps is that g_i^k is sos-concave, i.e., $-\nabla^2 g_i^k(x) = W(x)^T W(x)$ for some possibly nonsquare matrix polynomial $W(x)$. Also when ∂S contains singular points u we have additional conditions which are sufficient: for example, adding $-\nabla^2 g_i^k(u) \succ 0$ where $\nabla g_i^k(u) = 0$ to the hypotheses of the statement above guarantees SDP representability. We emphasize that our conditions here concern only the quasi-concavity properties of defining polynomials g_i^k on the boundary ∂S instead of on the whole set S. See Theorems 3.3, 3.4, 3.5 and 3.9 for details.

Third, consider the SDP representability of the convex hull of a compact nonconvex set $T = \cup_{k=1}^m T_k$. Here $T_k = \{x \in \mathbb{R}^n : f_1^k(x) \geq 0, \cdots, f_{m_k}^k(x) \geq 0\}$ are defined by polynomials $f_i^k(x)$. To obtain sufficient and necessary conditions, we find that the critical object is the *convex boundary* $\partial_c T$, the maximum subset of ∂T contained in $\partial \text{conv}(T)$. Our main result for $\partial_c T$ having everywhere nonsingular boundary is approximately:

Assume each T_k has nonempty interior near $\partial_c T$ and the defining polynomials f_i^k are nonsingular at every point $u \in \partial_c T \cap Z(f_i^k)$ (i.e. $\nabla f_i^k(u) \neq 0$). Then sufficient for $\text{conv}(T)$ to be SDP representable is: every $\partial_c T \cap Z(f_i^k)$ has positive curvature (i.e. f_i^k is strictly quasi-concave on $\partial_c T \cap Z(f_i^k)$), and necessary is: every irredundant $Z(f_i^k)$ has nonnegative curvature on ∂S (i.e. f_i^k is quasi-concave at u whenever f_i^k is irredundant at $u \in \partial_c T \cap Z(f_i^k)$).

This generalizes our second result (above) concerning SDP representability of compact convex semialgebraic sets. Also (just as before) we successfully weaken the hypothesis in several directions, which covers various cases of singularity. For example, one other sufficient condition allows f_i^k to be sos-concave. When T_k has empty interior, we prove that a condition called the *positive definite Lagrange Hessian (PDLH) condition* is sufficient. See Theorems 4.4, 4.5, 4.6, 4.7 and 4.8 for details.

Let us comment on the constructions of lifted LMIs. In this paper we analyze two different types of constructions. One is a fundamental moment type relaxation due to Lasserre-Parrilo which builds LMIs (discussed in §4), while the other is a localization technique introduced in this paper. The second result stated above is proved in two different ways, one of which gives a refined result:

Given a basic closed semialgebraic set $S = $ closure of $\{x \in \mathbb{R}^n : g_1(x) > 0, \cdots, g_m(x) > 0\}$ with nonempty interior. If S is convex and its boundary ∂S is positively curved and nonsingular, then there exists a certain set of defining polynomials for S for which a Lasserre-Parrilo type moment relaxation gives the lifted LMI for S.

See §5 for the proof. A very different construction of lifted LMI is also given in §4 using the localization technique plus a Lasserre-Parrilo type moment construction.

Notations and Outline The following notations will be used. A polynomial $p(x)$ is said to be a *sum of squares (SOS)* if $p(x) = w(x)^T w(x)$ for some column vector polynomial $w(x)$. A matrix polynomial $H(x)$ is said to be SOS if $H(x) = W(x)^T W(x)$ for some possibly nonsquare matrix polynomial $W(x)$. $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes the Euclidean space of n-dimensional space of real numbers, $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. $\Delta_m = \{\lambda \in \mathbb{R}^m_+ : \lambda_1 + \cdots + \lambda_m = 1\}$ is the standard simplex. For $x \in \mathbb{R}^n$, $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. $B(u, r)$ denotes the open ball $\{x \in \mathbb{R}^n : \|x - u\| < r\}$ and $\bar{B}(u, r)$ denotes the closed ball $\{x \in \mathbb{R}^n : \|x - u\| \le r\}$. For a given set W, $\overline{W}$ denotes the closure of W, and ∂W denotes its topological boundary. For a given matrix A, A^T denotes its transpose. I_n denotes the $n \times n$ identity matrix.

The paper is organized as follows. Section 2 discusses the SDP representation of the convex hull of union of SDP representable sets. Section 3 discusses the SDP representability of convex semialgebraic sets. Section 4 discusses the SDP representability of convex hulls of nonconvex semialgebraic sets. Section 5 presents a similar version of Theorem 3.3 and gives a different but more geometric proof based on results of [6]. Section 6 concludes this paper and makes a conjecture.

2. THE CONVEX HULL OF UNION OF SDP REPRESENTABLE SETS

It is obvious the intersection of SDP representable sets is also SDP representable, but the union might not be because the union may not be convex. However, the convex hull of the union of SDP representable sets is a convex semialgebraic set. Is it also SDP representable? This section will address this issue.

Let $W_1, \cdots, W_m \subset \mathbb{R}^n$ be convex sets. Then their Minkowski sum

$$W_1 + \cdots + W_m = \{x = x_1 + \cdots + x_m : x_1 \in W_1, \cdots, x_m \in W_m\}$$

is also a convex set. If every W_k is given by some lifted LMI, then a lifted LMI for $W_1 + \cdots + W_m$ can also be obtained immediately by definition. Usually the union of convex sets $W_1, \cdots, W_m$ is no longer convex, but its convex hull $\mathrm{conv}(\cup_{k=1}^m W_k)$ is convex again. Is $\mathrm{conv}(\cup_{k=1}^m W_k)$ SDP representable if every W_k is? We give a lemma first.

Lemma 2.1. *If W_k are all nonempty convex sets, then*

$$\mathrm{conv}(\bigcup_{k=1}^m W_k) = \bigcup_{\lambda \in \Delta_m} (\lambda_1 W_1 + \cdots + \lambda_m W_m)$$

where $\Delta_m = \{\lambda \in \mathbb{R}^m_+ : \lambda_1 + \cdots + \lambda_m = 1\}$ is the standard simplex.

Proof. This is a special case of Theorem 3.3 in Rockafellar [12]. □

Based on Lemma 2.1, given SDP representable sets $W_1, \cdots, W_m$, it is possible to obtain a SDP representation for the convex hull $\mathrm{conv}(\cup_{k=1}^m W_k)$ directly from the lifted LMIs of all W_k under rather weak conditions. This is summarized in the following theorem.

Theorem 2.2. *Let $W_1, \cdots, W_m$ be nonempty convex sets given by SDP representations*

$$W_k = \left\{x \in \mathbb{R}^n : \exists\, u^{(k)},\ A^{(k)} + \sum_{i=1}^n x_i B_i^{(k)} + \sum_{j=1}^{N_k} u_j^{(k)} C_j^{(k)} \succeq 0\right\}$$

for some symmetric matrices $A^{(k)}, B_i^{(k)}, C_j^{(k)}$. Define a new set

$$(2.1) \qquad \mathcal{C} = \left\{\sum_{k=1}^m x^{(k)} : \exists \lambda \in \Delta_m,\ \exists\, u^{(k)},\ \lambda_k A^{(k)} + \sum_{i=1}^n x_i^{(k)} B_i^{(k)} + \sum_{j=1}^{N_k} u_j^{(k)} C_j^{(k)} \succeq 0,\ 1 \le k \le m\right\}.$$

Then we have the inclusion

$$(2.2) \qquad conv(\bigcup_{k=1}^{m} W_k) \subseteq \mathcal{C}$$

and the equality

$$(2.3) \qquad \overline{\mathcal{C}} = \overline{conv(\bigcup_{k=1}^{m} W_k)}.$$

In addition, if every W_k is bounded, then

$$(2.4) \qquad \mathcal{C} = conv(\bigcup_{k=1}^{m} W_k).$$

Remark: When some W_k is unbounded, $\mathcal{C}$ and $\mathrm{conv}(\cup_{k=1}^{m} W_k)$ might not be equal, but they have the same interior, which is good enough for solving optimization problems over $\mathrm{conv}(\cup_{k=1}^{m} W_k)$.

Proof. First, by definition of $\mathcal{C}$, (2.2) is implied immediately by Lemma 2.1.

Second, we prove (2.3). By (2.2), it is sufficient to prove

$$\mathcal{C} \subseteq \overline{\mathrm{conv}(\bigcup_{k=1}^{m} W_k)}.$$

Let $x = x^{(1)} + \cdots + x^{(m)} \in \mathcal{C}$; then there exist $\lambda \in \Delta_m$ and $u^{(k)}$ such that

$$(2.5) \qquad \lambda_k A^{(k)} + \sum_{i=1}^{n} x_i^{(k)} B_i^{(k)} + \sum_{j=1}^{N_k} u_j^{(k)} C_j^{(k)} \succeq 0, \quad 1 \le k \le m.$$

Without loss of generality, assume $\lambda_1 = \cdots = \lambda_\ell = 0$ and $\lambda_{\ell+1}, \cdots, \lambda_m > 0$. Then for $k = \ell + 1, \cdots, m$, we have $\frac{1}{\lambda_k} x^{(k)} \in W_k$ and

$$x^{(\ell+1)} + \cdots + x^{(m)} = \lambda_{\ell+1} \frac{1}{\lambda_{\ell+1}} x^{(\ell+1)} + \cdots + \lambda_m \frac{1}{\lambda_m} x^{(m)} \in \mathrm{conv}(\bigcup_{k=1}^{m} W_k).$$

Since $W_k \ne \emptyset$, there exist $y^{(k)} \in W_k$ and $v^{(k)}$ such that

$$A^{(k)} + \sum_{i=1}^{n} y_i^{(k)} B_i^{(k)} + \sum_{j=1}^{N_k} v_j^{(k)} C_j^{(k)} \succeq 0.$$

For this and (2.5), for arbitrary $\epsilon > 0$ small enough, we have

$$(2.6) \qquad \epsilon A^{(k)} + \sum_{i=1}^{n} (x_i^{(k)} + \epsilon y_i^{(k)}) B_i^{(k)} + \sum_{j=1}^{N_k} (u_j^{(k)} + \epsilon v_j^{(k)}) C_j^{(k)} \succeq 0, \quad when \ 1 \le k \le \ell$$

$$(2.7) \qquad \frac{1 - \ell\epsilon}{1 + (m - \ell)\epsilon} \left\{ (\lambda_k + \epsilon) A^{(k)} + \sum_{i=1}^{n} (x_i^{(k)} + \epsilon y_i^{(k)}) B_i^{(k)} + \sum_{j=1}^{N_k} (u_j^{(k)} + \epsilon v_j^{(k)}) C_j^{(k)} \right\} \succeq 0, \quad \ell + 1 \le k \le m.$$

Now we let

$$x^{(k)}(\epsilon) := x^{(k)} + \epsilon y^{(k)} \ (1 \le k \le \ell), \qquad x^{(k)}(\epsilon) := \frac{1 - \ell\epsilon}{1 + (m - \ell)\epsilon} (x^{(k)} + \epsilon y^{(k)}) \ (\ell + 1 \le k \le m),$$

$$\lambda_k(\epsilon) = \epsilon \ (1 \le k \le \ell), \qquad \lambda_k(\epsilon) = \frac{1 - \ell\epsilon}{1 + (m - \ell)\epsilon} (\lambda_k + \epsilon) \ (\ell + 1 \le k \le m).$$

In this notation (2.6)(2.7) become

$$(2.8) \qquad \lambda_k(\epsilon) A^{(k)} + \sum_{i=1}^{n} x_i^{(k)}(\epsilon) B_i^{(k)} + \sum_{j=1}^{N_k} \tilde{u}_j^{(k)}(\epsilon) C_j^{(k)} \succeq 0, \quad 1 \le k \le m$$

with $\tilde{u}_j^{(k)}(\epsilon) := (u_j^{(k)} + \epsilon v_j^{(k)})$. Obviously $\lambda(\epsilon) \in \Delta_m$ and $0 < \lambda_k(\epsilon) < 1$ for every $1 \leq k \leq m$. From LMI (2.8) and from $\lambda_k(\epsilon) > 0$ we get $\frac{1}{\lambda_k(\epsilon)} x^{(k)}(\epsilon) \in W_k$ for all k. Let $x(\epsilon) := x^{(1)}(\epsilon) + \cdots + x^{(m)}(\epsilon)$; then we have

$$x(\epsilon) = \lambda_1(\epsilon) \frac{1}{\lambda_1(\epsilon)} x^{(1)}(\epsilon) + \cdots + \lambda_m(\epsilon) \frac{1}{\lambda_m(\epsilon)} x^{(m)}(\epsilon) \in \operatorname{conv}(\bigcup_{k=1}^m W_k).$$

As $\epsilon \to 0$, $x(\epsilon) \to x$, which implies $x \in \overline{\operatorname{conv}(\bigcup_{k=1}^m W_k)}$.

Third, we prove (2.4). When every W_k is bounded, it suffices to show $\mathcal{C} \subseteq \operatorname{conv}(\cup_{k=1}^m W_k)$. Suppose $x = x^{(1)} + \cdots + x^{(m)} \in \mathcal{C}$ with some $\lambda \in [0,1]$ and $u^{(k)}$. Without loss of generality, assume $\lambda_1 = \cdots = \lambda_\ell = 0$ and $\lambda_{\ell+1}, \cdots, \lambda_m > 0$. Obviously, for every $k = \ell + 1, \cdots, m$, we have $\frac{1}{\lambda_k} x^{(k)} \in W_k$ and

$$x^{(\ell+1)} + \cdots + x^{(m)} = \lambda_{\ell+1} \frac{1}{\lambda_{\ell+1}} x^{(\ell+1)} + \cdots + \lambda_m \frac{1}{\lambda_m} x^{(m)} \in \operatorname{conv}(\bigcup_{k=1}^m W_k).$$

Since $W_k \neq \emptyset$, there exist $y^{(k)}$ and $v^{(k)}$ such that

$$A^{(k)} + \sum_{i=1}^n y_i^{(k)} B_i^{(k)} + \sum_{j=1}^{N_k} v_j^{(k)} C_j^{(k)} \succeq 0.$$

Combining the above with (2.5) and observing $\lambda_1 = \cdots = \lambda_\ell = 0$, we obtain that

$$A^{(k)} + \sum_{i=1}^n (y_i^{(k)} + \alpha x_i^{(k)}) B_i^{(k)} + \sum_{j=1}^{N_k} (v_j^{(k)} + \alpha u_j^{(k)}) C_j^{(k)} \succeq 0, \quad \forall \alpha > 0, \ \forall \ 1 \leq k \leq \ell.$$

Hence, we must have $x^{(k)} = 0$ for $k = 1, \cdots, \ell$, because otherwise

$$y^{(k)} + [0, \infty) x^{(k)}$$

is an unbounded ray in W_k, which contradicts the boundedness of W_k. Thus

$$x = x^{(\ell+1)} + \cdots + x^{(m)} \in \operatorname{conv}(\bigcup_{k=1}^m W_k)$$

which completes the proof. $\square$

Example 2.3. When some W_k is unbounded, $\mathcal{C}$ and $\operatorname{conv}(\cup_{k=1}^m W_k)$ might not be equal, and $\mathcal{C}$ might not be closed. Let us see some examples.
(i) Consider $W_1 = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0 \right\}, W_2 = \{0\}$. The convex hull $\operatorname{conv}(W_1 \cup W_2) = \{x \in \mathbb{R}_+^2 : x_1 + x_2 = 0 \text{ or } x_1 x_2 > 0\}$. However,

$$\mathcal{C} = \left\{ x \in \mathbb{R}^2 : \exists \ 0 \leq \lambda_1 \leq 1, \begin{bmatrix} x_1 & \lambda_1 \\ \lambda_1 & x_2 \end{bmatrix} \succeq 0 \right\} = \mathbb{R}_+^2.$$

$\mathcal{C}$ and $\operatorname{conv}(W_1 \cup W_2)$ are not equal.
(ii) Consider $W_1 = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 & 1 + x_2 \\ 1 + x_2 & 1 + u \end{bmatrix} \succeq 0 \right\}$ and $W_2 = \{0\}$. We have $\operatorname{conv}(W_1 \cup W_2) = \{x \in \mathbb{R}^2 : x_1 > 0, \text{ or } x_1 = 0 \text{ and } -1 \leq x_2 \leq 0\}$ and $\overline{\operatorname{conv}(W_1 \cup W_2)} = \{x \in \mathbb{R}^2 : x_1 \geq 0\}$. But $\mathcal{C} = \operatorname{conv}(W_1 \cup W_2)$ is not closed.

Example 2.4. Now we see some examples showing that the boundedness of $W_1, \cdots, W_m$ is not necessary for (2.4) to hold.

(a) Consider the special case that each W_k is homogeneous, i.e., i.e., $A^{(k)} = 0$ in the SDP representation of W_k. Then by Lemma 2.1, we immediately have

$$\mathcal{C} = \text{conv}(\bigcup_{k=1}^{m} W_k).$$

(b) Consider $W_1 = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0 \right\}, W_2 = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0 \right\}$. It can be verified that $\text{conv}(W_1 \cup W_2)$ is given by

$$\mathcal{C} = \left\{ x + y \in \mathbb{R}^2 : \exists \lambda \in \Delta_2, \begin{bmatrix} -x_1 & \lambda_1 \\ \lambda_1 & x_2 \end{bmatrix} \succeq 0, \begin{bmatrix} y_1 & \lambda_2 \\ \lambda_2 & y_2 \end{bmatrix} \succeq 0 \right\} = \{ x \in \mathbb{R}^2 : x_2 > 0 \}.$$

3. Sufficient and necessary conditions for SDP representable sets

In this section, we present sufficient conditions and necessary conditions for SDP representability of a compact convex semialgebraic set S. As we will see, these sufficient conditions and necessary conditions are very close with the main gaps being between the boundary ∂S having positive versus nonnegative curvature and between the defining polynomials being singular or not on the part of the boundary where they vanish. A case which bypasses the gaps is when some defining polynomials are sos-concave, i.e., their negative Hessians are SOS.

Our approach is to start with convex sets which are basic semialgebraic, and to give weaker sufficient conditions than those given in [6]: the defining polynomials are either sos-concave or strictly quasi-concave on the part of the boundary ∂S where they vanish (not necessarily on the whole set). And then we give similar sufficient conditions for convex sets that are *not* basic semialgebraic. Lastly, we give necessary conditions for SDP representability: the defining polynomials are quasi-concave on nonsingular points on the part of the boundary of S where they vanish.

Let us begin with reviewing some background about curvature and quasi-concavity. The key technique for proving the sufficient conditions is to localize to small balls containing a piece of ∂S, use the strictly quasi-concave function results (Theorem 2 in [6]) to represent these small sets, and then to apply Theorem 2.2 to patch all of these representations together, thereby obtaining an SDP representation of S.

3.1. Curvature and quasi-concavity

We first review the definition of *curvature*. For a smooth function $f(x)$ on $\mathbb{R}^n$, suppose the zero set $Z(f) := \{ x \in \mathbb{R}^n : f(x) = 0 \}$ is nonsingular at a point $u \in Z(f)$, i.e., $\nabla f(u) \neq 0$. Then $Z(f)$ is a smooth hypersurface near the point u. $Z(f)$ is said to have *positive curvature* at the nonsingular point $u \in Z(f)$ if its *second fundamental form* is positive definite, i.e.,

(3.1) $$- v^T \nabla^2 f(u) v > 0, \quad \forall 0 \neq v \in \nabla f(u)^\perp$$

where $\nabla f(u)^\perp := \{ v \in \mathbb{R}^n : \nabla f(u)^T v = 0 \}$. For a subset $V \subset Z(f)$, we say $Z(f)$ has *positive curvature on V* if $f(x)$ is nonsingular on V and $Z(f)$ has positive curvature at every $u \in V$. When $>$ is replaced by $\geq$ in (3.1), we can similarly define $Z(f)$ has *nonnegative curvature* at u. We emphasize that this definition applies to any zero sets defined by smooth functions on their nonsingular points. This is needed in §5. We refer to Spivak [13] for more on curvature and the second fundamental form.

The sign "$-$" in the front of (3.1) might look confusing for some readers, since $Z(f)$ and $Z(-f)$ define exactly the same zero set. Geometrically, the curvature of a hypersurface should be independent of the sign of the defining functions. The reason for including the minus sign in (3.1) is we are interested in the case where the set $\{ x : f(x) \geq 0 \}$ is locally convex near u when $Z(f)$ has positive curvature at u. Now we give more geometric perspective by describing alternative formulations of positive curvature. Geometrically, the zero set $Z(f)$ has nonnegative (resp. positive) curvature at a nonsingular point $u \in Z(f)$ if and only if there exists an open set $\mathcal{O}_u$ such that $Z(f) \cap \mathcal{O}_u$ can be represented as the graph of a function ϕ which

is (strictly) convex at the origin in an *appropriate* coordinate system (see Ghomi [5]). Here we define a function to be convex (resp. strictly convex) at some point if its Hessian is positive semidefinite (resp. definite) at that point. Also note when $Z(f)$ has positive curvature at u, the set $\{x : f(x) \geq 0\}$ is locally convex near u if and only if (3.1) holds, or equivalently the set $\{x : f(x) \geq 0\} \cap \mathcal{O}_u$ is above the graph of function ϕ. Now we prove the statements above and show such ϕ exists. When the gradient $\nabla f(u) \neq 0$, by the Implicit Function Theorem, in an open set near u the hypersurface $Z(f)$ can be represented as the graph of some smooth function in a certain coordinate system. Suppose the origin of this coordinate system corresponds to the point u, and the set $\{x : f(x) \geq 0\}$ is locally convex near u. Let us make the affine linear coordinate transformation

$$(3.2) \qquad x - u = \begin{bmatrix} \nabla f(u) & G(u) \end{bmatrix}^T \begin{bmatrix} y \\ x' \end{bmatrix}$$

where $(y, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ are new coordinates and $G(u)$ is an orthogonal basis for subspace $\nabla f(u)^\perp$. By the Implicit Function Theorem, since $\nabla f(u) \neq 0$, in some neighborhood $\mathcal{O}_u$ of u, the equation $f(x) = 0$ defines a smooth function $y = \phi(x')$. For simplicity, we reuse the letter f and write $f(x', y) = f(x', \phi(x')) = 0$. Since $\nabla f(u)$ is orthogonal to $G(u)$, we have $f_y(0,0) = 1$ and $\nabla_{x'}\phi(0) = 0$. Twice differentiating $f(x', y) = 0$ gives

$$\nabla_{x'x'}f + \nabla_{x'y}f\nabla_{x'}\phi^T + \nabla_{x'}\phi\nabla_{x'y}f^T + f_{yy}\nabla_{x'}\phi\nabla_{x'}\phi^T + f_y\nabla_{x'x'}\phi = 0.$$

Evaluate the above at the origin in the new coordinates (y, x'), to get

$$\nabla_{x'x'}\phi(0) = -\nabla_{x'x'}f(u).$$

So we can see $Z(f)$ has positive (resp. nonnegative) curvature at u if and if the function $y = \phi(x')$ is strictly convex (resp. convex) at u. Since at u the direction $\nabla f(u)$ points to the inside of the set $\{x : f(x) \geq 0\}$, the intersection $\{x : f(x) \geq 0\} \cap \mathcal{O}_u$ lies above the graph of ϕ.

The notion of positive curvature of a nonsingular hypersurface $Z(f)$ does not distinguish one side of $Z(f)$ from the other. For example, the boundary of the unit ball $\bar{B}(0,1)$ is the unit sphere, a manifold with positive curvature by standard convention. However, $\bar{B}(0,1)$ can be expressed as $\{x : f(x) \geq 0\}$ where $f(x) = 1 - \|x\|^2$, or equivalently as $\{x : h(x) \leq 0\}$ where $h(x) = \|x\|^2 - 1$. Note that $Z(f) = Z(h)$, but $-\nabla^2 f(x) \succ 0$ and $+\nabla^2 h(x) \succ 0$.

However, on a nonsingular hypersurface $Z(f)$ one can designate its sides by choosing one of the two normal directions $\pm\nu(x)$ at points x on $Z(f)$. We call one such determination at some point, say u, the *outward direction*, and then select, at each x, the continuous function $\nu(x)$ to be consistent with this determination. In the ball example, $\nabla f(x) = -2x$ and we would typically choose $\nu(x) = -\nabla f(x)$ to be the outward normal direction to $Z(f)$. In the more general case described below equation (3.2), let us call $-\nabla f(x)$ the outward normal, which near the origin points away from the set $\{(x', y) : y \geq \phi(x')\}$. To see this, note that $-\nabla f(0,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

We remark that the definition of positive curvature for some hypersurface Z at a nonsingular point is independent of the choice of defining functions. Suppose f and g are smooth defining functions such that $Z \cap B(u, \delta) = Z(f) \cap B(u, \delta) = Z(g) \cap B(u, \delta), \nabla f(u) \neq 0 \neq \nabla g(u)$ for some $\delta > 0$ and

$$(3.3) \qquad \{x \in B(u, \delta) : f(x) \geq 0\} = \{x \in B(u, \delta) : g(x) \geq 0\}.$$

Then the second fundamental form in terms of f is positive definite (resp. semidefinite) at u if and only if the second fundamental form in terms of g is positive definite (resp. semidefinite) at u. To see this, note that $\nabla f(u) = \alpha \nabla g(u)$ for some scalar $\alpha \neq 0$, because $\nabla f(u)$ and $\nabla g(u)$ are perpendicular to the boundary of Z at u. Also $\alpha > 0$ because of (3.3). Then in the new coordinate system (y, x') defined in (3.2), as we have seen earlier, Z has nonnegative (resp. positive) curvature at u if and only if the function $y = \phi(x')$ is convex (resp. strictly convex) at u, which holds if and only if either one of f and g has positive definite (resp. semidefinite) second fundamental form. So the second fundamental form of f and g are simultaneously positive definite or semidefinite.

The smooth function $f(x)$ on $\mathbb{R}^n$ is said to be *strictly quasi-concave at u* if the condition (3.1) holds. When $\nabla f(u)$ vanishes, we require $-\nabla^2 f(u) \succ 0$ in order for $f(x)$ to be strictly quasi-concave at u. For a subset $V \subset \mathbb{R}^n$, we say $f(x)$ is *strictly quasi-concave on V* if $f(x)$ is strictly quasi-concave on every point on V. When $>$ is replaced by $\geq$ in (3.1), we can similarly define $f(x)$ to be *quasi-concave*. We remark that our definition of quasi-concavity here is slightly less demanding than the usual definition of quasi-concavity in the existing literature (see Section 3.4.3 in [2]).

Recall that a polynomial $g(x)$ is said to be *sos-concave* if $-\nabla^2 g(x) = W(x)^T W(x)$ for some possibly nonsquare matrix polynomial $W(x)$. The following theorem gives sufficient conditions for SDP representability in terms of sos-concavity or strict quasi-concavity.

Theorem 3.1. *(Theorem 2 [6]) Suppose $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is a compact convex set defined by polynomials $g_i(x)$ and has nonempty interior. For each i, if $g_i(x)$ is either sos-concave or strictly quasi-concave on S, then S is SDP representable.*

3.2. Sufficient and necessary conditions on defining polynomials

In this subsection, we give sufficient conditions as well as necessary conditions for SDP representability for both basic and nonbasic convex semialgebraic sets. These conditions are about the properties of defining polynomials on the part of the boundary where they vanish, instead of the whole set. This is different from the conditions given in [6]. Let us begin with a proposition which is often used later.

Proposition 3.2. *Let S be a compact convex set. Then S is SDP representable if and only if for every $u \in \partial S$, there exists some $\delta > 0$ such that $S \cap \bar{B}(u, \delta)$ is SDP representable.*

Proof. " $\Rightarrow$ " Suppose S has SDP representation

$$S = \left\{ x \in \mathbb{R}^n : A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{N} u_j C_j \succeq 0 \right\}$$

for symmetric matrices A, B_i, C_j. Then $S \cap \bar{B}(u, \delta)$ also has SDP representation

$$\left\{ x \in \mathbb{R}^n : A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{N} u_j C_j \succeq 0, \quad \begin{bmatrix} I_n & x - u \\ (x - u)^T & \delta^2 \end{bmatrix} \succeq 0 \right\}.$$

" $\Leftarrow$ " Suppose for every $u \in \partial S$ the set $S \cap \bar{B}(u, \delta_u)$ has SDP representation for some $\delta_u > 0$. Note that $\{B(u, \delta_u) : u \in \partial S\}$ is an open cover for the compact set ∂S. So there are a finite number of balls, say, $B(u_1, \delta_1), \cdots, B(u_L, \delta_L)$, to cover ∂S. Note that

$$S = \operatorname{conv}(\partial S) = \operatorname{conv}(\bigcup_{k=1}^{L} (\partial S \cap \bar{B}(u_k, \delta_k))) \subseteq \operatorname{conv}(\bigcup_{k=1}^{L} (S \cap \bar{B}(u_k, \delta_k))) \subseteq S.$$

The sets $S \cap \bar{B}(u_k, \delta_k)$ are all bounded. By Theorem 2.2, we know

$$S = \operatorname{conv}(\bigcup_{k=1}^{L} S \cap \bar{B}(u_k, \delta_k))$$

has SDP representation. $\square$

When the set S is basic closed semialgebraic, we have the following sufficient condition for SDP representability, which strengthens Theorem 3.1.

Theorem 3.3. *Assume $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is a compact convex set defined by polynomials g_i and has nonempty interior. If for every $u \in \partial S$ and i for which $g_i(u) = 0$, g_i is either sos-concave or strictly quasi-concave at u, then S is SDP representable.*

Remarks: (i) This result is stronger than Theorem 2 of [6] which requires each g_i is either *sos-concave* or strictly quasi-concave on the whole set S instead of only on the boundary. (ii) The special case that some of the g_i are linear is included in sos-concave case. (iii) Later we will present a slightly weaker version of Theorem 3.3 by using conditions on the curvature of the boundary and give a very different but more geometric proof based on Theorems 3 and 4 in [6]. This is left in §5.

Proof. For any $u \in \partial S$, let $I(u) = \{1 \leq i \leq m : g_i(u) = 0\}$. For every $i \in I(u)$, if $g_i(x)$ is not sos-concave, $g_i(x)$ is strictly quasi-concave at u. By continuity, there exist some $\delta > 0$ such that $g_i(x)$ is strictly quasi-concave on $\bar{B}(u, \delta)$. Note $g_i(u) > 0$ for $i \notin I(u)$. So we can choose $\delta > 0$ small enough such that

$$g_i(x) > 0, \forall\, i \notin I(u),\ \forall\, x \in \bar{B}(u, \delta).$$

Therefore, the set $S_u := S \cap \bar{B}(u, \delta)$ can be defined equivalently by only using active g_i, namely,

$$S_u = \left\{ x \in \mathbb{R}^n : g_i(x) \geq 0, \forall\, i \in I(u),\ \delta^2 - \|x - u\|^2 \geq 0 \right\}.$$

For every $i \in I(u)$, the defining polynomial $g_i(x)$ is either sos-concave or strictly quasi-concave on S_u. Obviously S_u is a compact convex set with nonempty interior. By Theorem 3.1, S_u is SDP representable. And hence by Proposition 3.2, S is also SDP representable. $\qquad\square$

Now we turn to the SDP representability problem when S is not basic semialgebraic. Assume $S = \bigcup_{k=1}^{m} T_k$ is compact convex. Here each $T_k = \{x \in \mathbb{R}^n : g_1^k(x) \geq 0, \cdots, g_m^k(x) \geq 0\}$ is basic closed semialgebraic but not necessarily convex. Similar sufficient conditions on T_k for the SDP representability of S can be established.

Theorem 3.4 (Sufficient conditions for SDP representability). *Suppose* $S = \bigcup_{k=1}^{m} T_k$ *is a compact convex semialgebraic set with each*

$$T_k = \{x \in \mathbb{R}^n : g_1^k(x) \geq 0,\ \cdots,\ g_{m_k}^k(x) \geq 0\}$$

being defined by polynomials $g_i^k(x)$. *If for every* $u \in \partial S$, *and each* g_i^k *satisfying* $g_i^k(u) = 0$, T_k *has interior near* u *and* $g_i^k(x)$ *is either sos-concave or strictly quasi-concave at* u, *then* S *is SDP representable.*

Proof. By Proposition 3.2, it suffices to show that for each $u \in \partial S$ there exists $\delta > 0$ such that the intersection $S \cap \bar{B}(u, \delta)$ is SDP representable. For each $u \in \partial S$, let $I_k(u) = \{1 \leq i \leq m_k : g_i^k(u) = 0\}$. By assumption, for every $i \in I_k(u)$, if $g_i^k(x)$ is not sos-concave, g_i^k is strictly quasi-concave at u. By continuity, g_i^k is strictly quasi-concave on $\bar{B}(u, \delta)$ for some $\delta > 0$. Note $g_i^k(u) > 0$ for all $i \notin I_k(u)$. So $\delta > 0$ can be chosen sufficiently small so that

$$g_i^k(x) > 0, \forall\, i \notin I(u),\ \forall\, x \in \bar{B}(u, \delta).$$

Then we can see

$$T_k \cap \bar{B}(u, \delta_u) = \left\{ x \in \mathbb{R}^n : g_i^k(x) \geq 0, \forall\, i \in I_k(u),\ \delta^2 - \|x - u\|^2 \geq 0 \right\}.$$

For every $i \in I_k(u)$, the defining polynomial $g_i^k(x)$ is either sos-concave or strictly quasi-concave on $T_k \cap \bar{B}(u, \delta_u)$. Hence, the intersection $T_k \cap \bar{B}(u, \delta_u)$ is a compact convex set with nonempty interior. By Theorem 3.1, $T_k \cap \bar{B}(u, \delta_u)$ is SDP representable. Therefore, by Theorem 2.2, we know

$$S \cap \bar{B}(u, \delta) = \operatorname{conv}(S \cap \bar{B}(u, \delta)) = \operatorname{conv}\Big(\bigcup_{k=1}^{m} T_k \cap \bar{B}(u, \delta) \Big) = \operatorname{conv}\Big(\bigcup_{k=1}^{m} \big(T_k \cap \bar{B}(u, \delta) \big) \Big)$$

is also SDP representable. $\qquad\square$

If the defining polynomials of a compact convex set S are either sos-concave or strictly quasi-concave on the part of the boundary of S where they vanish, Theorem 3.4 tell us S is SDP representable. If S is the convex hull of the union of such convex sets, Theorem 2.2 tells us that S is also SDP representable. We now assert that this is not very far from the necessary conditions for S to be SDP representable.

We now need give a short review of smoothness of the boundary of a set. Let $S = \bigcup_{k=1}^{m} T_k$ and $T_k = \{x \in \mathbb{R}^n : g_1^k(x) \geq 0, \cdots, g_{m_k}^k(x) \geq 0\}$ with ∂S and ∂T_k denoting their topological boundaries. For any $u \in \partial T_k(u)$, the active constraint set $I_k(u) = \{1 \leq i \leq m_k : g_i^k(u) = 0\}$ is nonempty.

We say u is a *nonsingular point* on ∂T_k if $|I_k(u)| = 1$ and $\nabla g_i^k(u) \neq 0$ for $i \in I_k(u)$. u is called a *corner point* on ∂T_k if $|I_k(u)| > 1$, and is nonsingular if $\nabla g_i^k(u) \neq 0$ for every $i \in I_k(u)$. For $u \in \partial S$ and $i \in I_k(u) \neq \emptyset$, we say the defining function g_i^k is *irredundant at u with respect to ∂S* (or just *irredundant at u* if the set S is clear from the context) if there exists a sequence of nonsingular points $\{u_N\} \subset Z(g_i^k) \cap \partial S$ such that $u_N \rightarrow u$; otherwise, we say g_i^k is *redundant* at u. We say g_i^k is *nonsingular* at u if $\nabla g_i^k(u) \neq 0$. Geometrically, when g_i^k is nonsingular at $u \in \partial S$, g_i^k being redundant at u means that the constraint $g_i^k(x) \geq 0$ could be removed without changing $S \cap B(u, \delta)$ for $\delta > 0$ small enough. A corner point $u \in \partial T_k$ is said to be *nondegenerate* if g_i^k is both irredundant and nonsingular at u whenever $i \in I_k(u) \neq \emptyset$.

The following gives necessary conditions for SDP representability.

Theorem 3.5. *(Necessary conditions for SDP representability) If the convex set S is SDP representable, then the following holds:*

(a) *The interior $\overset{\circ}{S}$ of S is a finite union of basic open semialgebraic sets, i.e.,*

$$\overset{\circ}{S} = \bigcup_{k=1}^{m} T_k, \quad T_k = \{x \in \mathbb{R}^n : g_1^k(x) > 0, \cdots, g_{m_k}^k(x) > 0\}$$

for some polynomials $g_i^k(x)$.

(b) *The closure $\overline{S}$ of S is a finite union of basic closed semialgebraic sets, i.e.,*

$$\overline{S} = \bigcup_{k=1}^{m} T_k, \quad T_k = \{x \in \mathbb{R}^n : g_1^k(x) \geq 0, \cdots, g_{m_k}^k(x) \geq 0\}$$

for some polynomials $g_i^k(x)$ (they might be different from those in (a) above).

(c) *For each $u \in \partial \overline{S}$ and $i \in I_k(u) \neq \emptyset$, if g_i^k from (b) is irredundant and nonsingular at u, then g_i^k is quasi-concave at u.*

Remarks: (i) The proof of Theorem 3.5 only depends on the fact that S is a convex semialgebraic set with nonempty interior, and does not use its SDP representation. (ii) The polynomials $g_i^k(x)$ in item (b) might be different from the polynomials $g_i^k(x)$ in item (a). We use the same notations for convenience.

Proof. (a) and (b) can be seen immediately from Theorem 2.7.2 in [3].

(c) Let $u \in \partial \overline{S} \cap \partial T_k$. Note that $\overline{S}$ is a convex set and has the same boundary as S.

First, consider the case that u is a smooth point. Since $\overline{S}$ is convex, $\partial \overline{S}$ has a supporting hyperplane $u + w^\perp = \{u + x : w^T x = 0\}$. $\overline{S}$ lies on one side of $u + w^\perp$ and so does T_k, since T_k is contained in $\overline{S}$. Since u is a smooth point, $I_k(u) = \{i\}$ has cardinality one. For some $\delta > 0$ sufficiently small, we have

$$T_k \cap B(u, \delta) = \{x \in \mathbb{R}^n : g_i^k(x) \geq 0, \delta^2 - \|x - u\|^2 > 0\}.$$

Note $u + w^\perp$ is also a supporting hyperplane of T_k passing through u. So, the gradient $\nabla g_i^k(u)$ must be parallel to w, i.e., $\nabla g_i^k(u) = \alpha_i^k w$ for some nonzero scalar $\alpha_i^k \neq 0$. Thus, for all $0 \neq v \in w^\perp$ and $\epsilon > 0$ small enough, the point $u + \frac{\epsilon}{\|v\|} v$ is not in the interior of $T_k \cap B(u, \delta)$, which implies

$$g_i^k(u + \frac{\epsilon}{\|v\|} v) \leq 0, \ \forall 0 \neq v \in w^\perp = \nabla g_i^k(u)^\perp.$$

By the second order Taylor expansion, we have

$$-v^T \nabla^2 g_i^k(u) v \geq 0, \ \forall 0 \neq v \in \nabla g_i^k(u)^\perp,$$

that is, g_i^k is quasi-concave at u.

Second, consider the case that $u \in \partial \overline{S}$ is a corner point. By assumption that g_i^k is irredundant and nonsingular at u, there exists a sequence of smooth points $\{u_N\} \subset Z(g_i^k) \cap \partial \overline{S}$ such that $u_N \rightarrow u$ and $\nabla g_i^k(u) \neq 0$.

So $\nabla g_i^k(u_N) \neq 0$ for N sufficiently large. From the above, we know that

$$-v^T \nabla^2 g_i^k(u_N) v \geq 0, \ \forall 0 \neq v \in \nabla g_i^k(u_N)^{\perp}.$$

Note that the subspace $\nabla g_i^k(u_N)^{\perp}$ equals the range space of the matrix $R(u_N)$ where

$$R(v) := I_n - \frac{1}{\left(g_i^k(v)\right)^T g_i^k(v)} g_i^k(v) \left(g_i^k(v)\right)^T.$$

So the quasi-concavity of g_i^k at u_N is equivalent to

$$-R(u_N)^T \nabla^2 g_i^k(u_N) R(u_N) \succeq 0.$$

Since $\nabla g_i^k(u) \neq 0$, we have $R(u_N) \to R(u)$ Therefore, letting $N \to \infty$, we get

$$-R(u)^T \nabla^2 g_i^k(u) R(u) \succeq 0,$$

which implies

$$-v^T \nabla^2 g_i^k(u) v \geq 0, \ \forall 0 \neq v \in \nabla g_i^k(u)^{\perp},$$

that is, g_i^k is quasi-concave at u. $\square$

We point out that in (c) of Theorem 3.5 the condition that g_i^k is irredundant can not be dropped. For a counterexample, consider the set

$$S = \left\{ x \in \mathbb{R}^2 : g_1^1(x) := 1 - x_1^2 - x_2^2 \geq 0, \ g_2^1(x) := (x_1 - 2)^2 + x_2^2 - 1 \geq 0 \right\}.$$

Choose $u = (1, 0)$ on the boundary. Then g_2^1 is redundant at u. As we can see, g_2^1 is not quasi-concave at u.

By comparing Theorem 3.4 and Theorem 3.5, we can see the presented sufficient conditions and necessary conditions are pretty close. The main gaps are between the defining polynomials being positive versus nonnegative curvature and between the defining polynomials being singular or not on the part of the boundary where they vanish. A case which bypasses the gaps is when some defining polynomials are sos-concave.

As is obvious, the set of defining polynomials for a semialgebraic set is not unique, e.g., the set remains the same if each defining polynomial is replaced by its cubic power. However, as we can imagine, if we use some set of defining polynomials, we can prove the SDP representability of the set, but if we use some other set of defining of polynomials, we might not be able to prove that. A simple example is that the set $\{x : g(x) := (1 - \|x\|^2)^3 \geq 0\}$ is obviously SDP representable but none of our earlier theorems using $g(x)$ only can show this set is SDP representable. This is because, so far, we have discussed the SDP representability only from the view of the defining polynomials, instead of from the view of the geometric properties of the convex sets. Sometimes, we are more interested in the conditions on the geometry of convex sets which is independent of defining polynomials. This leads us to the next subsection of giving conditions on the geometric properties.

3.3. Sufficient and necessary conditions on the geometry

In this subsection, to address the SDP representability of convex semialgebraic sets, we give sufficient conditions and necessary conditions on the geometry of the sets instead of on their defining polynomials.

A subset $V \subset \mathbb{R}^n$ is a *variety* if there exist polynomials $p_1(x), \cdots, p_m(x)$ such that $V = \{x \in \mathbb{R}^n : p_1(x) = \cdots = p_m(x) = 0\}$. Given a variety V, define the ideal $I(V)$ as

$$I(V) = \{p \in \mathbb{R}[x] : p(u) = 0 \text{ whenever } u \in V\}.$$

Let the ideal $I(V)$ be generated by polynomials $q_1, \cdots, q_k$. A point $u \in V$ is said to be a *nonsingular point* if the matrix $\left[\frac{\partial q_i}{\partial x_j}(u)\right]$ has full rank. V is said to be a *nonsingular variety* if every point of V is a nonsingular point. Note that if two varieties V_1, V_2 are both nonsingular at a certain point u, then

their intersection variety $V_1 \cap V_2$ might be singular at u. A set $Z \subset \mathbb{R}^n$ is said to be *Zariski open* if its complement in $\mathbb{R}^n$ is a variety. We refer to [3, 4] for more on algebraic varieties.

Lemma 3.6. *Let $S \subset \mathbb{R}^n$ be a compact convex semialgebraic set with nonempty interior. Then*

(i) *The interior $\overset{\circ}{S}$ is the union of basic open semialgebraic sets, i.e.,*

$$\overset{\circ}{S} = \bigcup_{k=1}^m T_k, \quad T_k := \{x \in \mathbb{R}^n : g_1^k(x) > 0, \cdots, g_{m_k}^k(x) > 0\}$$

where $g_i^k(x)$ are polynomials. Each T_k is bounded and its closure has boundary ∂T_k.

(ii) *The Zariski closure $\mathcal{V}^k$ of each ∂T_k is the union $\mathcal{V}^k = \mathcal{V}_1^k \cup \mathcal{V}_2^k \cup \cdots \cup \mathcal{V}_{L_k}^k$ of irreducible varieties of dimension $n-1$ such that $\mathcal{V}_i^k \cap \partial T_k \not\subseteq (\cup_{j \neq i} \mathcal{V}_i^k) \cap \partial T_k$. We can write these as $\mathcal{V}_i^k = \{x \in \mathbb{R}^n : f_i^k(x) = 0\}$ for some irreducible polynomials $f_i^k(x)$ such that the ideal $I(\mathcal{V}_i^k)$ is generated by $f_i^k(x)$. Furthermore, if every $\mathcal{V}_i^k$ containing $u \in \partial T_k$ is nonsingular at u (i.e., $\nabla f_i^k(u) \neq 0$), then for $r > 0$ sufficiently small we have*

(3.4) $$\overline{T_k} \cap \bar{B}(u, r) = \big(\bigcap_{1 \leq i \leq L_k} \mathcal{V}_i^{k+} \big) \cap \bar{B}(u, r), \qquad \mathcal{V}_i^{k+} := \{x \in \mathbb{R}^n : f_i^k(x) \geq 0\}.$$

(iii) *For $u \in \partial S \cap \partial T_k \cap \mathcal{V}_i^k$, we say $\mathcal{V}_i^k$ is irredundant at u if there exists a sequence $\{u_N\} \subset \partial S$ converging to u such that $S \cap B(u_N, \epsilon_N) = \mathcal{V}_i^{k+} \cap B(u_N, \epsilon_N)$ for some $\epsilon_N > 0$. If $\mathcal{V}_i^k$ is nonsingular and irredundant at u, then the curvature of $\mathcal{V}_i$ at u is nonnegative.*

(iv) *The nonsingular points in $\mathcal{V}_i^k$ form a Zariski open subset of $\mathcal{V}_i^k$ and their complement has a lower dimension than $\mathcal{V}_i^k$ does.*

In the above lemma, the irreducible varieties $\mathcal{V}_i^k$ are called the *intrinsic varieties* of ∂S, and the corresponding polynomials f_i^k are called the *intrinsic polynomials* of S. Note that every $\mathcal{V}_i^k$ is a hypersurface. The intrinsic $\mathcal{V}_i^k$ is called *irredundant* if it is irredundant at every $u \in \partial S \cap \partial T_k \cap \mathcal{V}_i^k$. $\mathcal{V}_i^k$ is called *redundant* at $u \in \partial S \cap \partial T_k \cap \mathcal{V}_i^k$ if it is not irredundant at u. The set $\mathcal{B} = \{\mathcal{V}_i^k : 1 \leq k \leq m, 1 \leq i \leq L_k\}$ of irreducible varieties in (ii) above is called a *boundary sheet* of S. We remark that the boundary sheet $\mathcal{B}$ of S is not unique.

Example 3.7. Consider the compact convex set $S = \{x \in \bar{B}(0,1) : x_2 \geq x_1^2 \text{ or } x \in \mathbb{R}_+^2\}$. Define irreducible varieties $\mathcal{V}_i^k$ as follows

$$\mathcal{V}_1^1 = \{x \in \mathbb{R}^n : 1 - \|x\|^2 = 0\}, \quad \mathcal{V}_2^1 = \{x \in \mathbb{R}^n : x_2 - x_1^2 = 0\}$$

$$\mathcal{V}_1^2 = \mathcal{V}_1^1, \quad \mathcal{V}_2^2 = \{x \in \mathbb{R}^n : x_2 = 0\}, \quad \mathcal{V}_3^2(a) = \{x \in \mathbb{R}^n : x_1 - ax_2^2 = 0\} \ (0 \leq a \leq 1).$$

They are the intrinsic varieties of ∂S. For any $0 \leq a \leq 1$, $\mathcal{B}(a) = \{\mathcal{V}_1^1, \mathcal{V}_2^1, \mathcal{V}_1^2, \mathcal{V}_2^2, \mathcal{V}_3^2(a)\}$ is a boundary sheet of S. It is not unique. $\mathcal{V}_1^1, \mathcal{V}_2^1, \mathcal{V}_1^2, \mathcal{V}_2^2$ are all irredundant, while $\mathcal{V}_3^2(a)$ is redundant at the origin.

Proof of Lemma 3.6. Note that S is the closure of its interior. Pick any point $u \in \partial S$ and pick an interior point o to S. The interior points of the interval joining o to u must lie in the interior of S and can approach its vertex u.

(i) This is the claim (a) of Theorem 3.5.

(ii) T_k is a component of

$$\check{T}_k := \{x : g^k(x) > 0\}$$

where $g^k := g_1^k g_2^k \cdots g_{m_k}^k$ and is what [7] calls an algebraic interior. In other words, any bounded basic open semialgebraic set is an algebraic interior. Lemma 2.1 of [7] now tells us that a minimum degree defining polynomial $\tilde{g}^k$ for T_k is unique up to a multiplicative constant. Also it says that any other defining polynomial h for T_k equals $p\tilde{g}^k$ for some polynomial p. Thus $\tilde{g}^k(v) = 0$ and $\nabla \tilde{g}^k(v) = 0$ implies $\nabla h(v) = 0$. So the singular points of h on ∂T_k contain the singular points of ∂T_k. Lemma 2.1 of [7] characterizes the boundary of algebraic interiors. The third and fourth paragraphs in the proof of Lemma 2.1 of [7] show that the Zariski closure of ∂T_k is a union of irreducible varieties $\mathcal{V}_i^k$ each of dimension

$n-1$ which satisfy all requirements of (ii) except equation (3.4). Without loss of generality, the sign of f_i^k can be chosen such that $f_i^k(x)$ is nonnegative on T_k. When every $\mathcal{V}_i^k$ is nonsingular at $u \in \partial T_k \cap \mathcal{V}_i^k$, there exists $r > 0$ small enough such that every $\mathcal{V}_i^k$ is a smooth hypersurface on $\bar{B}(u, r)$. So on $\bar{B}(u, r)$, a point v is on the boundary of T_k if and only if all $f_i^k(v) \geq 0$ and at least one $f_i^k(v) = 0$; on the other hand, v is in the interior of T_k if and only if all $f_i^k(v) > 0$. Therefore equation (3.4) holds.

(iii) This is implied by item (c) of Theorem 3.5.

(iv) The $\mathcal{V}_i$ above are irreducible algebraic varieties. Thus by Proposition 3.3.14 of [3] the desired conclusions on the nonsingular points follows.   □

In terms of intrinsic varieties, our main result about SDP representability is

Theorem 3.8. *Let S be a compact convex semialgebraic set with nonempty interior, and $\mathcal{B} = \{\mathcal{V}_i^k : 1 \leq k \leq m, 1 \leq i \leq L_k\}$ be a boundary sheet of S as guaranteed by Lemma 3.6. Assume every hypersurface $\mathcal{V}_i^k$ in $\mathcal{B}$ is nonsingular on $\mathcal{V}_i^k \cap \partial S$, and has positive curvature at $u \in \mathcal{V}_i^k \cap \partial S$ whenever $\mathcal{V}_i^k$ is redundant at u. Then S is SDP representable if (resp. only if) for each $u \in \partial S \cap \mathcal{V}_i^k$ the hypersurface $\mathcal{V}_i^k$ has positive (resp. nonnegative) curvature at u.*

Proof. The necessary side is (iii) of Lemma 3.6. Let us prove the sufficient side. By Proposition 3.2, it suffices to show that for every $u \in \partial S$ there exists $\delta > 0$ such that $S \cap \bar{B}(u, \delta)$ is SDP representable. Let $\mathcal{V}_i^k$ and f_i^k be given by Lemma 3.6. Fix an arbitrary point $u \in \partial S$ and let $I_k(u) = \{1 \leq i \leq L_k : u \in \mathcal{V}_i^k\}$. By the assumption of nonsingularity of $\mathcal{V}_i^k$ on $\mathcal{V}_i^k \cap \partial S$ and equation (3.4) in Lemma 3.6, there is some $\delta > 0$ small enough such that

$$S \cap \bar{B}(u, \delta) = \bigcup_{k=1}^m \overline{T_k} \cap \bar{B}(u, \delta)$$

$$\overline{T_k} \cap \bar{B}(u, \delta) = \left\{ x \in \mathbb{R}^n : f_i^k(x) \geq 0, \, \forall i \in I_k(u), \, \delta^2 - \|x - u\|^2 \geq 0 \right\}.$$

Note that f_i^k are irreducible polynomials and nonsingular (their gradients do not vanish) on $\mathcal{V}_i^k \cap \partial S$. So the positive curvature hypothesis implies that each $f_i^k(x)$ is strictly quasi-concave on $\bar{B}(u, \delta)$ (we can choose $\delta > 0$ small enough to make this true). Obviously $T_k \cap \bar{B}(u, \delta)$ is a bounded set. By Theorem 3.1 and Theorem 2.2, we know $S \cap \bar{B}(u, \delta)$ is SDP representable.   □

In terms of intrinsic polynomials, the above theorem can be reformulated as

Theorem 3.9. *Let S be a compact convex semialgebraic set with nonempty interior, and f_i^k ($1 \leq k \leq m, 1 \leq i \leq L_k$) be intrinsic polynomials of S as guaranteed by Lemma 3.6. Assume every f_i^k is nonsingular on $Z(f_i^k) \cap \partial S$, and strict quasi-concave at $u \in Z(f_i^k) \cap \partial S$ whenever f_i^k is redundant at u. Then S is SDP representable if (resp. only if) for each $u \in \partial S$ and f_i^k satisfying $f_i^k(u) = 0$ the intrinsic polynomial f_i^k is strictly quasi-concave (resp. non-strictly quasi-concave) at u.*

Remarks: (i) In the above two theorems, we assume intrinsic varieties (resp. intrinsic polynomials) are positively curved (resp. strictly quasi-concave) on the part of the boundary where they are redundant. This assumption is reasonable, because redundant intrinsic varieties (resp. intrinsic polynomials) are usually not unique and there is a freedom of choosing them. (ii) As mentioned in the introduction, under the nonsingularity assumption, the gap between sufficient and necessary conditions is the intrinsic varieties being positively curved versus nonnegatively curved or the intrinsic polynomials being strictly quasi-concave versus nonstrictly quasi-concave. A case bypassing the gap is the intrinsic polynomials being sos-concave, as shown in Theorem 3.4. Thus, in Example 3.7, we know the compact set there is SDP representable. (iii) In Theorems 3.8 and 3.9, to prove the necessary conditions, we have only used the convexity of S and its nonempty interior, instead of the SDP representability of S. Thus the necessary conditions in Theorems 3.8 and 3.9 are still true when S is a convex semialgebraic set with nonempty interior.

4. Convex hulls of nonconvex semialgebraic sets

In this section, we consider the problem of finding the convex hull of a nonconvex semialgebraic set T. The convex hull $\text{conv}(T)$ must be convex and semialgebraic (Theorem 2.2.1 in [3]). By Theorem 2.7.2 in [3], the closure of $\text{conv}(T)$ is a union of basic closed semialgebraic sets. A fundamental problem in convex geometry and semidefinite programming is to find the SDP representation of $\text{conv}(T)$. This section will address this problem and prove the sufficient conditions and necessary conditions for the SDP representability of $\text{conv}(T)$ summarized in the Introduction.

Let T be a compact nonconvex set with boundary ∂T. Obviously $\text{conv}(T)$ is the convex hull of the boundary ∂T. Some part of ∂T might be in the interior of $\text{conv}(T)$ and will not contribute to $\text{conv}(T)$. So we are motivated to define the *convex boundary* $\partial_c T$ of T as

$$(4.1) \qquad \partial_c T = \left\{ u \in T : \ell^T u = \min_{x \in T} \ell^T x \text{ for some } \ell \in \mathbb{R}^n \text{ with } \|\ell\| = 1 \right\} \subseteq \partial T.$$

Geometrically, $\partial_c T$ is the maximum subset of ∂T contained in $\partial \text{conv}(T)$, and the convex hull of $\partial_c T$ is still $\text{conv}(T)$.

Proposition 4.1. *If T is compact, then $\text{conv}(\partial_c T) = \text{conv}(T)$ and $\partial_c T$ is also compact.*

Proof. Obviously $\text{conv}(\partial_c(T)) \subseteq \text{conv}(T)$. We need to prove $\text{conv}(\partial_c(T)) \supseteq \text{conv}(T)$. It suffices to show that if $u \notin \text{conv}(\partial_c T)$ then $u \notin \text{conv}(T)$. For any $u \notin \text{conv}(\partial_c T)$, by the Convex Set Separation Theorem, there is a vector ℓ of unit length and a positive number $\delta > 0$ such that

$$\ell^T u < \ell^T x - \delta, \quad \forall\, x \in \text{conv}(\partial_c T).$$

Let $v \in T$ minimize $\ell^T x$ over T, which must exist due to the compactness of T. Then $v \in \partial_c T$ and hence

$$\ell^T u < \ell^T v - \delta = \min_{x \in T} \ell^T x - \delta.$$

Therefore, $u \notin \text{conv}(T)$.

Clearly $\partial_c T$ is bounded and closed by its definition. So $\partial_c T$ is compact. □

Remark: If T is not compact, then Proposition 4.1 might not be true. For instance, for set $T = \{x \in \mathbb{R}^2 : \|x\|^2 \geq 1\}$, the convex boundary $\partial_c T = \emptyset$, but $\text{conv}(T)$ is the whole space. When T is not compact, even if $\text{conv}(\partial T) = \text{conv}(T)$, it is still possible that $\text{conv}(\partial_c T) \neq \text{conv}(T)$. As a counterexample, consider the set

$$W = \{(0,0)\} \cup \{x \in \mathbb{R}_+^2 : x_1 x_2 \geq 1\}.$$

It can be verified that $\text{conv}(W) = \text{conv}(\partial W)$, $\partial_c W = \{(0,0)\}$ and $\text{conv}(\partial_c W) \neq \text{conv}(W)$.

Note that every semialgebraic set is a finite union of basic semialgebraic sets (Proposition 2.1.8 in [3]). To find the convex hull of a semialgebraic set T, by Theorem 2.2, it suffices to find the SDP representation of the convex hull of each basic semialgebraic subset of T.

Theorem 4.2. *Let $T_1, \cdots, T_m$ be bounded semialgebraic sets. If each $\text{conv}(T_k)$ is SDP representable, then the convex hull of $\cup_{k=1}^m T_k$ is also SDP representable.*

Proof. By Theorem 2.2, it suffices to prove that

$$\text{conv}\left(\bigcup_{k=1}^m T_k\right) = \text{conv}\left(\bigcup_{k=1}^m \text{conv}(T_k)\right).$$

Obviously, the left hand side is contained in the right hand side. We only prove the converse. For every $j = 1, \ldots, m$, we have

$$\text{conv}(T_j) \subseteq \text{conv}\left(\bigcup_{k=1}^m T_k\right).$$

Now taking the union of left hand side for $j = 1, \ldots, m$, we get

$$\bigcup_{j=1}^{m} \mathrm{conv}(T_j) \subseteq \mathrm{conv}(\bigcup_{k=1}^{m} T_k).$$

Taking the convex hull of the above on both sides results in

$$\mathrm{conv}(\bigcup_{j=1}^{m} \mathrm{conv}(T_j)) \subseteq \mathrm{conv}(\bigcup_{k=1}^{m} T_k),$$

which implies the equality at the beginning of this proof. $\square$

Proposition 4.3. *Let T be a compact semialgebraic set. Then $\mathrm{conv}(T)$ is SDP representable if for every $u \in \partial_c T$, there exists $\delta > 0$ such that $\mathrm{conv}(T \cap \bar{B}(u, \delta))$ is SDP representable.*

Proof. Suppose for every $u \in \partial_c T$ the set $\mathrm{conv}(T \cap \bar{B}(u, \delta_u))$ has SDP representation for some $\delta_u > 0$. Note that $\{B(u, \delta_u) : u \in \partial_c T\}$ is an open cover of the compact set $\partial_c T$. So there are a finite number of balls, say, $B(u_1, \delta_1), \cdots, B(u_L, \delta_L)$, to cover $\partial_c T$. Noting

$$\mathrm{conv}(\partial_c T) \subseteq \mathrm{conv}(\bigcup_{k=1}^{L} \partial_c T \cap \bar{B}(u_k, \delta_k)) \subseteq \mathrm{conv}(\bigcup_{k=1}^{L} \mathrm{conv}(T \cap \bar{B}(u_k, \delta_k))) \subseteq \mathrm{conv}(T),$$

by Proposition 4.1, we have

$$\mathrm{conv}(T) = \mathrm{conv}(\bigcup_{k=1}^{L} \mathrm{conv}(T \cap \bar{B}(u_k, \delta_k))).$$

The sets $\mathrm{conv}(T \cap \bar{B}(u_k, \delta_k))$ are all bounded. By Theorem 2.2, we know $\mathrm{conv}(T)$ is SDP representable. $\square$

Remark: By Proposition 4.3, to find the SDP representation of the convex hull of a compact set T, it is sufficient to find the SDP representations of convex hulls of the intersections of T and small balls near the convex boundary $\partial_c T$. This gives the bridge between the global and local SDP representations of convex hulls.

In the following two subsections, we prove some sufficient conditions and necessary conditions for the SDP representability of convex hulls. They are essentially generalizations of Section 3 and the results in [6].

4.1. Sos-concavity or quasi-concavity conditions

In Section 3, we have proven some sufficient conditions and necessary conditions for the SDP representability of compact convex sets. In this subsection, we prove similar conditions for the convex hulls of nonconvex sets. Throughout this subsection, consider the semialgebraic sets which have nonempty interior (then there are no equality defining polynomials). We begin with basic semialgebraic sets, and then consider more general semialgebraic sets.

Theorem 4.4. *Assume $T = \{x \in \mathbb{R}^n : f_1(x) \geq 0, \cdots, f_m(x) \geq 0\}$ is a compact set defined by polynomials $f_i(x)$ and has nonempty interior near $\partial_c T$, i.e., for every $u \in \partial_c T$ and $\delta > 0$ small enough, there exists $v \in B(u, \delta)$ such that $f_i(v) > 0$ for all $i = 1, \ldots, m$. If for each $u \in \partial_c T$ and i for which $f_i(u) = 0$, $f_i(x)$ is either sos-concave or strictly quasi-concave at u, then $\mathrm{conv}(T)$ is SDP representable.*

Proof. By Proposition 4.3, we only need prove for every $u \in \partial_c T$ the set $\mathrm{conv}(T \cap \bar{B}(u, \delta))$ is SDP representable for some $\delta > 0$. For an arbitrary $u \in \partial_c T$, and let $I(u) = \{1 \leq i \leq m : f_i(u) = 0\}$. For any $i \in I(u)$, if $f_i(x)$ is not sos-concave, f_i is strictly quasi-concave at u. By continuity, f_i is strictly quasi-concave on $\bar{B}(u, \delta)$ for some $\delta > 0$. Note $f_i(u) > 0$ for all $i \notin I(u)$. Therefore, by continuity, the number $\delta > 0$ can be chosen small enough that $f_i(x) > 0$ for all $x \in \bar{B}(u, \delta)$ and $i \notin I(u)$. Then we can see

$$T_u := T \cap \bar{B}(u, \delta) = \{x \in \mathbb{R}^n : f_i(x) \geq 0, \, \forall \, i \in I(u), \quad \delta^2 - \|x - u\|^2 \geq 0\}.$$

For every $i \in I(u)$, the polynomial $f_i(x)$ is either sos-concave or strictly quasi-concave on T_u. Clearly, T_u is a compact convex set with nonempty interior. By Theorem 3.1, we know $\mathrm{conv}(T_u) = T_u$ is SDP representable, since T_u is convex. $\qquad\square$

Now we consider nonbasic semialgebraic sets and give similar sufficient conditions.

Theorem 4.5 (Sufficient conditions for SDP representability of convex hulls). *Assume $T = \bigcup_{k=1}^{L} T_k$ is a compact semialgebraic set with*

$$T_k = \{x \in \mathbb{R}^n : f_1^k(x) \geq 0, \cdots, f_{m_k}^k(x) \geq 0\}$$

being defined by polynomials $f_i^k(x)$. If for each $u \in \partial_c T$ and f_i^k for which $f_i^k(u) = 0$, T_k has interior near u and f_i^k is either sos-concave or strictly quasi-concave at u, then $\mathrm{conv}(T)$ is SDP representable.

Proof. By Proposition 4.3, it suffices to prove for each $u \in \partial_c T$, there exists $\delta > 0$ such that $\mathrm{conv}(T \cap \bar{B}(u, \delta))$ is SDP representable. Fix an arbitrary $u \in \partial_c T$, and let $I_k(u) = \{1 \leq i \leq m_k : f_i^k(u) = 0\}$. By assumption, if $i \in I_k(u)$ and $f_i^k(x)$ is not sos-concave, f_i^k is strictly quasi-concave at u. Thus, by continuity, there exists $\delta > 0$ so that f_i^k is strictly quasi-concave on $\bar{B}(u, \delta)$. Note that $f_i^k(u) > 0$ for all $i \notin I_k(u)$. So $\delta > 0$ can be chosen small enough such that $f_i^k(x) > 0$ for all $x \in \bar{B}(u, \delta)$ and $i \notin I_k(u)$. Then we can see that

$$T_k \cap \bar{B}(u, \delta_u) = \left\{x \in \mathbb{R}^n : f_i^k(x) \geq 0, \forall\, i \in I_k(u), \delta_u^2 - \|x - u\|^2 \geq 0\right\}$$

is a compact convex set with nonempty interior. And, for every $i \in I_k(u)$, $f_i^k(x)$ is either sos-concave or strictly quasi-concave on $\bar{B}(u, \delta)$. By Theorem 3.1, the set $T_k \cap \bar{B}(u, \delta_u)$ is SDP representable. By Theorem 2.2,

$$\mathrm{conv}(T \cap \bar{B}(u, \delta)) = \mathrm{conv}\left(\bigcup_{k=1}^{L} T_k \cap \bar{B}(u, \delta)\right)$$

is also SDP representable. $\qquad\square$

As in Theorem 3.5, we can get similar necessary conditions on the defining polynomials of the nonconvex sets.

Theorem 4.6 (Necessary conditions for SDP representability of convex hulls). *Assume $T = \bigcup_{k=1}^{L} T_k$ is a compact semialgebraic set with*

$$T_k = \{x \in \mathbb{R}^n : f_1^k(x) \geq 0, \cdots, f_{m_k}^k(x) \geq 0\}$$

being defined by polynomials $f_i^k(x)$, and assume its convex hull $\mathrm{conv}(T)$ is SDP representable. For each $u \in \partial_c T$ and $i \in I_k(u) \neq \emptyset$, if f_i^k is nonsingular and irredundant at u with respect to $\partial\mathrm{conv}(T)$, then f_i^k is quasi-concave at u.

Proof. Note that the convex hull $\mathrm{conv}(T)$ is compact and $T \subset \mathrm{conv}(T)$. By Theorem 2.7.2 of [3], there exist basic closed semialgebraic sets $T_{L+1}, \ldots, T_M$ such that

$$\mathrm{conv}(T) = \bigcup_{k=1}^{M} T_k.$$

Every T_k for $k = L + 1, \ldots, M$ can also be defined in the form

$$T_k = \{x \in \mathbb{R}^n : f_1^k(x) \geq 0, \cdots, f_{m_k}^k(x) \geq 0\}$$

for certain polynomials $f_i^k(x)$. The sets $T_1, \ldots, T_L$ are basic closed semialgebraic subsets of $\mathrm{conv}(T)$ and $\partial_c T \subseteq \partial\mathrm{conv}(T)$. Consider $\mathrm{conv}(T)$ as the set $\overline{S}$ in Theorem 3.5. Then the conclusion of this theorem is a direct application of item (c) of Theorem 3.5. $\qquad\square$

4.2. The PDLH condition

In the previous subsection, the nonconvex semialgebraic sets are assumed to have nonempty interior near the convex boundary $\partial_c T$, and so there can be no equality defining polynomials. Now, in this subsection, we consider the more general nonconvex semialgebraic sets which might have empty interior and equality defining polynomials. Then the sufficient conditions in the preceding subsection do not hold anymore. We need another kind of sufficient condition: *the positive definite Lagrange Hessian (PDLH) condition.* As in earlier sections, begin with basic semialgebraic sets.

Assume T is a compact basic semialgebraic set of the form

$$T = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_{m_1}(x) = 0, h_1(x) \geq 0, \cdots, h_{m_2}(x) \geq 0\}.$$

Let ∂T be the boundary of T. For $u \in \partial T$, we say T satisfies the *positive definite Lagrange Hessian (PDLH) condition* at u if there exists $\delta_u > 0$ such that, for every unit length vector $\ell \in \mathbb{R}^n$ and every $0 < \delta \leq \delta_u$, the first order optimality condition holds at any global minimizer for the optimization problem

(4.2)
$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \ell^T x \\ \text{s.t.} \quad & f_1(x) = \cdots = f_{m_1}(x) = 0 \\ & h_1(x) \geq 0, \cdots, h_{m_2}(x) \geq 0 \\ & \delta^2 - \|x - u\|^2 \geq 0 \end{aligned}$$

and the Hessian of the associated Lagrange function is positive definite on the ball $\bar{B}(u, \delta)$. To be more precise, let $m = m_2 + 1$ and $h_m(x) = \delta^2 - \|x - u\|^2$. The associated Lagrange function of (4.2) is

$$\mathcal{L}(x) = \ell^T x - \sum_{i=1}^{m_1} \lambda_i f_i(x) - \sum_{j=1}^{m} \mu_j h_j(x)$$

where $\mu_1 \geq 0, \cdots, \mu_m \geq 0$. Let v be a global minimizer of problem (4.2). Then the PDLH condition requires

$$\ell = \sum_{i=1}^{m_1} \lambda_i \nabla f_i(v) + \sum_{j=1}^{m} \mu_j \nabla h_j(v)$$

for some λ_i and $\mu_j \geq 0$, and the Hessian of the Lagrange function satisfies

$$\nabla^2 \mathcal{L}(x) = -\sum_{i=1}^{m_1} \lambda_i \nabla^2 f_i(x) - \sum_{j=1}^{m} \mu_j \nabla^2 h_j(x) \succ 0, \ \forall \, x \in \bar{B}(u, \delta).$$

Remark: The defined PDLH condition here is stronger than the PDLH condition defined in [6]. This is because the PDLH condition in [6] is defined for convex sets described by concave functions. However, in this paper, the set T here is nonconvex. We need stronger assumptions.

The next theorem is an extension of Theorem 1.1 in [6] to give sufficient conditions assuring the SDP representability of conv(T).

Theorem 4.7. *Let $T = \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_{m_1}(x) = 0, h_1(x) \geq 0, \cdots, h_{m_2}(x) \geq 0\}$ be a compact set defined by polynomials. If the PDLH condition holds at every $u \in \partial_c T$, then conv(T) is SDP representable.*

Proof. By Proposition 4.3, we only need prove for every $u \in \partial_c T$, there exists $\delta > 0$ such that conv$(T \cap \bar{B}(u, \delta))$ is SDP representable. Let $\delta = \delta_u > 0$ be given by the PDLH condition and define $T_u = T \cap \bar{B}(u, \delta)$. We now prove conv$(T_u)$ is SDP representable.

First, we construct the lifted LMI for T_u. Let $m = m_2 + 1$ and $h_m(x) = \delta^2 - \|x - u\|^2$. For integer N, define the monomial vector

$$[x^N] = \begin{bmatrix} 1 & x_1 & \cdots & x_n & x_1^2 & x_1 x_2 & \cdots & x_n^N \end{bmatrix}^T.$$

Define new polynomials $h^\nu(x) = h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x)$, where $\nu = (\nu_1, \cdots, \nu_m) \in \mathbb{Z}_+^m$. Let $d_\nu = \lceil \deg(h_1^{\nu_1} \cdots h_r^{\nu_m})/2 \rceil$ and $d_k = \lceil \deg(f_k)/2 \rceil$. For a fixed integer $N \geq d_\nu, d_k$, define

$$M_{N-d_\nu}(h^\nu y) = \int_{\mathbb{R}^n} h^\nu(x)[x^{N-d_\nu}][x^{N-d_\nu}]^T \, d\mu(x) = \sum_{0 \leq |\alpha| \leq 2N} A_\alpha^\nu y_\alpha$$

$$f_k^T y = \int_{\mathbb{R}^n} f_k(x) \, d\mu(x) = \sum_{0 \leq |\alpha| \leq 2d_k} f_\alpha^k y_\alpha.$$

Here $\mu(\cdot)$ can be any nonnegative measure such that $\mu(\mathbb{R}^n) = 1$, $y_\alpha = \int_{\mathbb{R}^n} x^\alpha d\mu(x)$ are the moments, A_α^ν are symmetric matrices, and f_α^k are scalars such that

$$h^\nu(x)[x^{N-d_\nu}][x^{N-d_\nu}]^T = \sum_{0 \leq |\alpha| \leq 2N} A_\alpha^\nu x^\alpha$$

$$f_k(x) = \sum_{0 \leq |\alpha| \leq 2d_k} f_\alpha^k x^\alpha.$$

If $\mathrm{supp}(\mu) \subseteq T$, then we have $y_0 = 1$ and

$$\left. \begin{array}{ll} \forall \, \nu \in \{0,1\}^m, & M_{N-d_\nu}(h^\nu y) \succeq 0 \\ \forall \, 1 \leq k \leq m, & f_k^T y = 0 \end{array} \right\}.$$

Let e_i denote the standard i-th unit vector in $\mathbb{R}^n$. If we set $y_0 = 1$ and $y_{e_i} = x_i$ in the above LMI, then it becomes the LMI

(4.3)
$$\left. \begin{array}{ll} \forall \, \nu \in \{0,1\}^m, & A_0^\nu + \sum_{1 \leq i \leq n} A_{e_i}^\nu x_i + \sum_{1 < |\alpha| \leq 2N} A_\alpha^\nu y_\alpha \succeq 0 \\ \forall \, 1 \leq k \leq m, & f_0^k + \sum_{1 \leq i \leq n} f_{e_i}^k x_i + \sum_{1 < |\alpha| \leq 2d_k} f_\alpha^k y_\alpha = 0 \end{array} \right\}.$$

Obviously, the projection of LMI (4.3) to x-space contains $\mathrm{conv}(T_u)$.

Second, we prove that every linear polynomial nonnegative on T_u has an SOS representation with uniform degree bound. Given any $\ell \in \mathbb{R}^n$ with $\|\ell\| = 1$, let ℓ^* be the minimum value of $\ell^T x$ over T_u and $v \in T_u$ be a global minimizer. By the PDLH condition, there exist Lagrange multipliers $\lambda_1, \cdots, \lambda_{m_1}$ and $\mu_1 \geq 0, \cdots, \mu_m \geq 0$ such that

$$\ell = \sum_{i=1}^{m_1} \lambda_i \nabla f_i(v) + \sum_{j=1}^{m} \mu_j \nabla h_j(v)$$

and the Hessian of the Lagrange function satisfies

$$\nabla^2 \mathcal{L}(x) = -\sum_{i=1}^{m_1} \lambda_i \nabla^2 f_i(x) - \sum_{j=1}^{m} \mu_j \nabla^2 h_j(x) \succ 0, \ \forall \, x \in \bar{B}(u, \delta).$$

Since the Lagrange multipliers λ_i and μ_j are continuous functions of ℓ on the unit sphere, there must exist constants $M > \epsilon > 0$ such that for all $x \in B(u, \delta)$

$$MI_n \succeq \int_0^1 \int_0^t \nabla^2 \mathcal{L}(v + s(x - v)) ds \, dt \succeq \epsilon I_n.$$

By Theorem 27 in [6], there exist SOS matrix polynomials $G_\nu(x)$ such that

$$\int_0^1 \int_0^t \nabla^2 \mathcal{L}(v + s(x - v)) ds \, dt = \sum_{\nu \in \{0,1\}^m} h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x) G_\nu(x)$$

and the degrees of summand polynomials are bounded by

$$\deg(h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x) G_\nu(x)) \leq \Omega(\frac{M}{\epsilon}).$$

Here $\Omega(\cdot)$ is a function depending on T_u. Let $f_\ell(x) = \mathcal{L}(x) - \ell^*$. Then $f_\ell(v) = 0$ and $\nabla f_\ell(v) = 0$. By Taylor expansion, we have

$$f_\ell(x) = (x-v)^T \left(\int_0^1 \int_0^t \nabla^2 \mathcal{L}(v + s(x-v)) ds\, dt \right) (x-v)$$

$$= \sum_{\nu \in \{0,1\}^m} \phi_\nu(x) h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x)$$

where $\phi_\nu(x) = (x-v)^T G_\nu(x)(x-v)$ are SOS scalar polynomials. Since $\mu_j \geq 0$, let

$$\sigma_\nu(x) = \phi_\nu(x) + \begin{cases} \mu_j & \text{if } \nu = e_j \\ 0 & \text{otherwise} \end{cases}$$

be new SOS polynomials. Then we have

$$\ell^T x - \ell^* = \sum_{k=1}^{m_1} \lambda_k f_k(x) + \sum_{\nu \in \{0,1\}^m} \sigma_\nu(x) h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x).$$

There is a uniform bound N independent of ℓ such that

(4.4) $\deg(f_k(x)), \deg(h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x)\sigma_\nu(x)) \leq 2N.$

Third, we will show that (4.3) is an SDP representation for $\operatorname{conv}(T_u)$ when N is given by (4.4). In the above, we have actually shown that a property called *Schmüdgen's Bounded Degree Nonnegative Representation (S-BDNR)* (see Helton and Nie [6]) holds, i.e., every affine polynomials $\ell^T x - \ell^*$ nonnegative on T belongs to the preordering generated by the $f_i's$ and $h_j's$ with uniform degree bounds. This implies a weaker property called the *Schmüdgen's Bounded Degree Representation (S-BDR)* (see Lasserre [8]) holds, i.e., almost every affine polynomials $\ell^T x - \ell^*$ positive on T belongs to the preordering generated by the $f_i's$ and $h_j's$ with uniform degree bounds. So Theorem 2 in [8] can be applied to show that the LMI (4.3) is a SDP representation of $\operatorname{conv}(T)$. For the convenience of readers, we give the direct proof here. Since the projection of (4.3) to x-space contains $\operatorname{conv}(T_u)$, it is sufficient to prove the converse. In pursuit of a contradiction, suppose there exists a vector $(\hat{x}, \hat{y})$ satisfying (4.3) such that $\hat{x} \notin \operatorname{conv}(T_u)$. By the Hahn-Banach Separation Theorem, there must exist a unit length vector ℓ such that

(4.5)
$$\begin{aligned} \ell^T \hat{x} < \ell^* = \min \quad & \ell^T x \\ \text{s.t.} \quad & f_1(x) = \cdots f_{m_1}(x) = 0 \\ & h_1(x) \geq 0, \cdots, h_m(x) \geq 0. \end{aligned}$$

Let v be the minimizer of $\ell^T x$ on T_u; of course $v \in \partial T_u$. By the PDLH condition, there exist Lagrange multipliers $\lambda_1, \cdots, \lambda_{m_1}$ and $\mu_1, \cdots, \mu_m \geq 0$ such that

$$\ell = \sum_{i=1}^{m_1} \lambda_i \nabla f_i(v) + \sum_{j=1}^m \mu_j \nabla h_i(v), \qquad \mu_j h_j(v) = 0, \quad \forall\, j = 1, \cdots, m.$$

As we have proved earlier, the identity

$$\ell^T x - \ell^* = \sum_{k=1}^{m_1} \lambda_k f_k(x) + \sum_{\nu \in \{0,1\}^m} \sigma_\nu(x) h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x)$$

holds for some SOS polynomials $\sigma_\nu(x)$ with uniform degree bound

$$\deg(\sigma_\nu(x) h_1^{\nu_1}(x) \cdots h_m^{\nu_m}(x)) \leq 2N.$$

Thus we can write $\sigma_\nu(x) = [x^{N-d_\nu}]^T W_\nu [x^{N-d_\nu}]$ for some symmetric positive semidefinite matrix $W_\nu \succeq 0$. In the above identity, replace each monomial x^α with $|\alpha| > 1$ by $\hat{y}_\alpha$, then we get, for $\hat{y}_0 = 1$ and every

$\hat{y}_{e_i} = \hat{x}_i, \dots,$

$$\ell^T \hat{x} - \ell^* = \sum_{k=1}^{m_1} \lambda_k \left(\sum_{0 \le |\alpha| \le 2d_k} f_\alpha^k \hat{y}_\alpha \right) + \sum_{\nu \in \{0,1\}^m} Trace \left(W_\nu \cdot \left(\sum_{0 \le |\alpha| \le 2N} A_\alpha^i \hat{y}_\alpha \right) \right) \ge 0,$$

which contradicts (4.5). $\hfill\square$

Theorem 4.8. *Let $T = \bigcup_{k=1}^L T_k$ be a compact semialgebraic set where*

$$T_k = \{x \in \mathbb{R}^n : f_{k,1}(x) = \cdots = f_{k,m_{k,1}}(x) = 0, h_{k,1}(x) \ge 0, \cdots, h_{k,m_{k,2}}(x) \ge 0\}.$$

If for each T_k, the PDLH condition holds at every $u \in \partial_c T \cap \partial T_k$, then $conv(T)$ is SDP representable.

Proof. By Proposition 4.3, it suffices to prove for each $u \in \partial_c T$, $conv(T \cap \bar{B}(u, \delta))$ is SDP representable for some $\delta > 0$. Fix an arbitrary $u \in \partial_c T$. Let $I(u) = \{1 \le k \le L : u \in \partial T_k\}$. Then, by assumption, the PDLH condition holds at u for every T_k with $k \in I(u)$, and thus the radius $\delta > 0$ required in the PDLH condition can be chosen uniformly for all $k \in I(u)$ since $I(u)$ is finite. Hence we have

$$\mathrm{conv}(T \cap \bar{B}(u, \delta)) = \mathrm{conv}(\bigcup_{k \in I(u)} T_k \cap \bar{B}(u, \delta)) = \mathrm{conv}(\bigcup_{k \in I(u)} \mathrm{conv}(T_k \cap \bar{B}(u, \delta))).$$

By the proof of Theorem 4.7, the set $\mathrm{conv}(T_k \cap \bar{B}(u, \delta))$ is SDP representable. Therefore, by Theorem 2.2, $\mathrm{conv}(T \cap \bar{B}(u, \delta))$ is also SDP representable. $\hfill\square$

Remark: It should be mentioned that the PDLH condition is a very strong condition. It requires that, when every linear functional is minimized over the nonconvex set $T \cap \bar{B}(u, \delta)$, the first order KKT condition holds and that the Hessian of the Lagrangian is positive definite at the minimizer. This might restrict the applications of Theorem 4.8 in some cases.

5. A MORE GEOMETRIC PROOF OF THEOREM 3.3

For which set S does there exist a set of defining polynomials for which the Lasserre-Parrilo type moment relaxations produce an SDP representation of S? The major challenge is that while S may be presented to us by polynomials for which the Lasserre-Parrilo type constructions fail, there might exist another set of defining polynomials for which such a construction succeeds. This requires us to be able to find a set of defining polynomials such that the Lasserre-Parrilo type constructions work.

This section presents a very different approach to proving a similar version of Theorem 3.3, since what we did there used the localization technique heavily. We shall show here that the Lasserre-Parrilo type moment construction gives an SDP representation by using a certain set of defining polynomials. The proof we shall give, based on Theorems 3 and 4 of Helton and Nie [6] and on the proof of a proposition of Ghomi [5] (on smoothing boundaries of convex sets), is also very geometrical.

For the convex set $S = \{x \in \mathbb{R}^n : g_1(x) \ge 0, \cdots, g_m(x) \ge 0\}$, define $S_i = \{x \in \mathbb{R}^n : g_i(x) \ge 0\}$ and $Z_i = \{x \in \mathbb{R}^n : g_i(x) = 0\}$. The zero set Z_i is a hypersurface. Suppose Z_i does not intersect the interior of S. Then $Z_i \cap S = Z_i \cap \partial S$ and so is contained in the boundary of S.

In addition to the definition of positive curvature, we need a hypothesis about the shape of $Z_i \cap \partial S$. We say $Z_i \cap \partial S$ has *strictly convex shape with respect to S* if there exists a relative open subset $Y_i \subset Z_i$ containing $Z_i \cap \partial S$ such that for every $p \in \overline{Y}_i$ the set $S \cup \overline{Y}_i$ lies in one side of the tangent plane $T_p(Z_i)$ of Z_i at p, and does not touch $T_p(Z_i)$ except p, that is, $T_p(Z_i) \cap (S \cup \overline{Y}_i) \subseteq \{p\}$. The notion of strictly convex shape follows the notion of *strictly convex hypersurface* introduced in Ghomi [5].

Theorem 5.1. *Let $S = \{x \in \mathbb{R}^n : g_1(x) \ge 0, \cdots, g_m(x) \ge 0\}$ be a compact convex set defined by polynomials g_i and assume S has nonempty interior. Assume $g_i(x) > 0$ whenever x is in the interior of S, $\nabla g_i(u) \ne 0$ whenever $u \in Z_i \cap S$, and $Z_i \cap \partial S$ has strictly convex shape with respect to S when $g_i(x)$ is not sos-concave. If for each $u \in \partial S$ and every i such that $g_i(u) = 0$ we have either g_i is sos-concave or Z_i has positive curvature at u, then S is SDP representable. Moreover, there is a certain set of defining*

polynomials for S for which the Lasserre-Parrilo moment construction (5.4) and (5.6) given in [6] *gives an SDP representation.*

5.1. Background from [6]

First we review some results of [6] with slight modification of notation used in the original version. For a smooth function $f(x)$, the set $\{x \in \mathbb{R}^n : f(x) \geq 0\}$ is called *poscurv-convex* if it is compact convex, and its boundary ∂T equals $Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$ which is smooth ($\nabla f(x)$ does not vanish on ∂T) and positively curved at every point $u \in Z(f)$. When $f(x)$ is restricted to be a polynomial, the set $\{x \in \mathbb{R}^n : f(x) \geq 0\}$ is said to be *sos-convex* if $f(x)$ is sos-concave.

Theorem 5.2. *(Theorem 3* [6]*) Given polynomials g_i, suppose $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is compact convex and has nonempty interior. If each $S_i := \{x \in \mathbb{R}^n : g_i(x) \geq 0\}$ is either sos-convex or poscurv-convex, then S is SDP representable.*

We now turn to more general cases. Recall that $Z_i = \{x \in \mathbb{R}^n : g_i(x) = 0\}$. We say $S_i = \{x \in \mathbb{R}^n : g_i(x) \geq 0\}$ is *extendable poscurv-convex with respect to S* if $g_i(x) > 0$ whenever x lies in the interior of S_i and there exists a poscurv-convex set $T_i = \{x : f_i(x) \geq 0\} \supseteq S$ such that $\partial T_i \cap S = \partial S_i \cap S$. In other words, $Z_i \cap \partial S$ can be extended to become the boundary of a poscurv-convex set. Note that the condition of extendable poscurv-convexity of S_i requires Z_i does not intersect the interior of S.

Theorem 5.3. *(Theorem 4* [6]*) Given polynomials g_i, suppose $S = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is compact convex and has nonempty interior. If each S_i is either sos-convex or extendable poscurv-convex with respect to S, then S is SDP representable.*

We re-emphasize that the proofs of these theorems in [6] provide a new set of defining polynomials for S (possibly bigger than the original set) for which the Lasserre-Parrilo type moment constructions (5.4) and (5.6) given in [6] also produce SDP representations of S.

Comparing Theorems 5.3 and 5.1, we can see that Theorem 5.3 implies Theorem 5.1 if we can show S_i *is extendable poscurv-convex with respect to S provided Z_i has positive curvature on S.* The main task of this section is to prove this point and what is new to the proof is mostly in the facts about convex sets which we now turn to.

5.2. Smoothing boundaries of convex sets

We begin with some notations. Let $T_p(M)$ denote the tangent plane at p to a smooth hypersurface M without boundary. Sometimes we need the tangent plane on a hypersurface $\overline{M}$ with boundary, but this will not be a problem for us, because $\overline{M}$ encountered in this section will be always contained in another smooth hypersurface $\tilde{M}$ without boundary. In this case, we still use the notation $T_p(\overline{M})$ rather than $T_p(\tilde{M})$. For a point $p \in \mathbb{R}^n$ and a set $B \subset \mathbb{R}^n$, define the distance

$$dist(p, B) = \inf\{\|p - b\|_2 : \quad b \in B\}.$$

For convex set S, the set $Z_i = \{x \in \mathbb{R}^n : g_i(x) = 0\}$ is a hypersurface in $\mathbb{R}^n$ and is smooth in a relatively open subset containing $Z_i \cap \partial S = Z_i \cap S$ by the nonsingularity of $Z_i \cap \partial S$. Suppose $U \subset Z_i$ is relatively open and $Z_i \cap \partial S \subset U$. Let $\nu : \overline{U} \to \mathbb{S}^{n-1}$ be the Gauss map, the map given by the unit outward normal. We determine the outward normal direction as follows. The smooth positively curved hypersurface $Z_i \cap \partial S$ has at each point p a unique direction $\pm\nu(p)$ perpendicular to its tangent plane. The convex set S lies in one side of the tangent planes of $\partial S \cap Z_i$. We select the $+\nu(p)$ for $p \in \partial S \cap Z_i$ to be pointed away from S and call this the outward direction. The outward direction is uniquely determined by the continuity of $\nu(p)$ on $\overline{U}$. Under this determination of outward normal direction, for any $p \in \overline{U}$, we say a set G lies to the inside (resp. outside) of the tangent plane $T_p(\overline{U})$ if $\langle q - p, \nu(p) \rangle \leq 0$ (resp. $\langle q - p, \nu(p) \rangle \geq 0$) for all $q \in G$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in Euclidean spaces.

The next lemma insures the extendability property of "pieces" of the boundary of a convex set.

Lemma 5.4. *Suppose S is convex compact. Fix an index i. Assume $\nabla g_i(u)$ is nonzero for every $u \in Z_i \cap \partial S$, the curvature of Z_i is positive at all u there, and Z_i does not intersect the interior of S. If $Z_i \cap \partial S$ has strictly convex shape with respect to S, then S_i is extendable poscurv-convex with respect to S, i.e., there exists a convex set T such that*

(i) *The boundary ∂T is nonsingular (so is smooth) and has positive curvature everywhere.*
(ii) *T is compact, $S \subset T$ and $Z_i \cap \partial S = \partial T \cap \partial S = \partial T \cap S$.*

Proof of Lemma 5.4: The proof we shall give is very similar to the proof of Proposition 3.3 in Ghomi [5]. We need construct a set T satisfying the conclusions of Lemma 5.4. But our construction of T is slightly different from the one given in [5]. We proceed the proof by showing Claims A,B,C,D and E.

Claim A There exists a relatively open subset $U \subset Z_i$ satisfying

(1) $Z_i \cap \partial S \subset U$;
(2) the closure $\overline{U}$ is compact;
(3) U is smooth and $\overline{U}$ has positive curvature everywhere;
(4) $S \cap U = \partial S \cap U = Z_i \cap \partial S$;
(5) the relative boundary $\partial \overline{U} := \overline{U} \smallsetminus U$ satisfies $\partial \overline{U} \cap S = \emptyset$;
(6) for any $p \in \overline{U}$, the set $S \cup \overline{U}$ lies strictly to the inside of $T_p(\overline{U})$, that is, it lies to the inside of $T_p(\overline{U})$ and $(S \cup \overline{U}) \cap \big(T_p(\overline{U}) \backslash \{p\}\big) = \emptyset$.

Proof. We show that the set $U = \{x \in Z_i : dist(x, Z_i \cap \partial S) < \epsilon\}$ satisfies all the conditions of Claim A when $\epsilon > 0$ is sufficiently small. Items (1), (2) are obvious. Since $\nabla g_i(x)$ does not vanish on $\partial S \cap Z_i$, it also does not vanish on in U when $\epsilon > 0$ is sufficiently small. From the algebraic definition of positive curvature in (3.1), we also know $\overline{U}$ has positive curvature when $\epsilon > 0$ is small. So item (3) is also true.

For item (4), we know that (1) implies
$$Z_i \cap \partial S \subset \partial S \cap U \subset S \cap U.$$
To prove they are all equal to each other, it suffices to show $S \cap U \subset Z_i \cap \partial S$. For any $a \in S \cap U$, the point a must belong to $Z_i \cap \partial S$, because otherwise Z_i intersects the interior of S, which contradicts an assumption of Lemma 5.4. So $S \cap U \subset Z_i \cap \partial S$ and then (4) holds.

For item (5), note that $\partial \overline{U} = \{x \in Z_i : dist(x, Z_i \cap \partial S) = \epsilon\}$. If $\partial \overline{U}$ intersects S, then there exists $a \in \partial \overline{U} \cap S$ such that $a \in Z_i$ and $dist(a, Z_i \cap \partial S) = \epsilon > 0$. Hence $a \notin \partial S$ and a must belong to the interior of S, which contradicts an assumption of Lemma 5.4. So (5) holds.

Item (6) is just from the condition that $Z_i \cap S$ has strictly convex shape with respect to S. □

Fix a relatively open set U satisfying Claim A. For any small t, define
$$U_t := \{p_t := p - t\nu(p)|p \in U\}.$$
By continuity, its closure is
$$\overline{U}_t := \{p_t := p - t\nu(p)|p \in \overline{U}\}.$$
Note that $U_0 = U$ and $\overline{U}_0 = \overline{U}$. Let $\partial \overline{U}_t$ be the relative boundary of $\overline{U}_t$, that is, $\partial \overline{U}_t = \overline{U}_t \backslash U_t$. Then for t small it holds that
$$\partial \overline{U}_t := \{p_t := p - t\nu(p)|\ \ p \in \partial \overline{U}\}.$$
Clearly, $\partial \overline{U} \cap S \subseteq \partial \overline{U} \cap (S \cap Z_i) = \emptyset$ as $S \cap Z_i \subset U$, hence
$$dist(\partial \overline{U}, S) := \min_{p \in \partial \overline{U}} dist(p, S) > 0$$
as both S and $\partial \overline{U}$ are compact. By $\partial \overline{U} \cap S = \emptyset$ (condition (5) of Claim A) and continuity of $\partial \overline{U}_t$, we have

(5.1) $$\partial \overline{U}_t \cap S = \emptyset \quad \forall\, t \in (-r, r)$$

for all $r > 0$ small enough.

Now we give some elementary geometric facts about $\overline{U}$ and $\overline{U}_t$.

Claim B For $r > 0$ sufficiently small, we have

  (i) U_r is smooth and $\overline{U}_r$ has positive curvature everywhere;
  (ii) $\overline{U}_r$ globally lies to the inside of the tangent plane $T_{p_r}(\overline{U}_r)$ at any $p_r \in \overline{U}_r$;
  (iii) $\nu(p_r) = \nu(p)$ for all $p \in \overline{U}$;
  (iv) for every $p \in \overline{U}$, $dist(p, \overline{U}_r) = dist(p, T_{p_r}(\overline{U}_r)) = r$.

Proof. Items (i)-(ii) are the conclusions of paragraph 1 in the proof of Proposition 3.3 [5]. So we refer to [5] for the proof.

(iii) This is a basic fact in differential geometry, but we include a proof here since it is brief. The hypersurface Z_i has a relatively open smooth subset $\widetilde{U} \supset \overline{U}$. Similarly as before, we define

$$\widetilde{U}_t := \{p_t := p - t\nu(p) | p \in \widetilde{U}\}.$$

Fix an arbitrary point $p \in \overline{U} \subset \widetilde{U}$. Let $\{\phi(t) : t \in \mathbb{R}\} \subset \widetilde{U}$ be an arbitrary smooth curve passing through p, say, $\phi(0) = p$. Since $\nu(p)$ is the normal to $\widetilde{U}$ at p, we have $\langle \nu(p), \phi'(0) \rangle = 0$. Then $\{\phi(t) - r\nu(\phi(t)) : t \in \mathbb{R}\} \subset \widetilde{U}_r$ is a smooth curve passing through p_r. The unit length condition $\|\nu(\phi(t))\|_2^2 = 1$ of normals implies

$$\langle \nu(\phi(t)), \nabla_\phi \nu(\phi(t))\phi'(t) \rangle = 0, \forall t.$$

In particular, $\langle \nu(\phi(0)), \nabla_\phi \nu(\phi(0))\phi'(0) \rangle = 0$. Thus we have

$$\left\langle \nu(p), \frac{d(\phi(t) - r\nu(\phi(t)))}{dt}\Big|_{t=0} \right\rangle = \langle \nu(p), \phi'(0) \rangle - r\langle \nu(\phi(0)), \nabla_\phi \nu(\phi(0))\phi'(0) \rangle = 0.$$

So the curve $\{\phi(t) - r\nu(\phi(t)) : t \in \mathbb{R}\}$ in $\widetilde{U}_r$ is also perpendicular to $\nu(p)$. By uniqueness of unit normals of smooth hypersurfaces, we have $\nu(p_r) = \nu(p)$.

(iv) For every $p \in \overline{U}$, (iii) says $\nu(p_r) = \nu(p)$. So the point p lies to the outside of the tangent plane $T_{p_r}(\overline{U}_r)$. Since $p = p_r + r\nu(p_r)$ and $\nu(p_r)$ is perpendicular to $T_{p_r}(\overline{U}_r)$ at p_r, we have $r = dist(p, T_{p_r}(\overline{U}_r))$. From (ii), we know that $\overline{U}_r$ lies to the inside of the tangent plane $T_{p_r}(\overline{U}_r)$. So

$$dist(p, \overline{U}_r) \geq dist(p, T_{p_r}(\overline{U}_r)) = r.$$

Since $p_r = p - r\nu(p) \in U_r$, we obtain $dist(p, \overline{U}_r) \leq r$. Therefore, we have $dist(p, \overline{U}_r) = dist(p, T_{p_r}(\overline{U}_r)) = r$. $\qquad\square$

Claim C For any $q \in \overline{U}_r$, the set $S \smallsetminus (\cup_{0 \leq t < r} U_t)$ globally lies to the inside of $T_{q_r}(\hat{U}_r)$ when r is sufficiently small.

Proof. We prove this claim in three steps.

Step 1 From item (ii) of Claim B we know the set $\overline{U}_s$ lies to the inside of all the tangent planes of $\overline{U}_s$ when $s > 0$ is small enough. For every $q_t \in \overline{U}_t$, the tangent plane $T_{q_t}(\overline{U}_t)$ always lies to the inside of the tangent plane $T_{q_s}(\overline{U}_s)$ when $0 \leq s \leq t$ are both small. This is because $q_s = q_t + (t - s)\nu(q_t)$, since $\nu(q_s) = \nu(q_t)$ from item (iii) of Claim B. Hence for $\delta > 0$ small enough, the set $\overline{U}_t$ lies to the inside of all the tangent planes of $\overline{U}_s$ whenever $0 \leq s \leq t \leq \delta$.

Step 2 Fix a $\delta > 0$ sufficiently small as required in Step 1. Define the set

$$W_\delta = S \smallsetminus (\cup_{0 \leq t < \delta} U_t).$$

For $\eta > 0$ sufficiently small, it holds that

(5.2) $$W_\delta = S \smallsetminus (\cup_{-\eta < t < \delta} U_t).$$

This is because U_t for $t \in (-\eta, 0)$ lies outside of S, due to item (6) of Claim A and item (iii) of Claim B.

Next, we show that the set $U_{(-\eta,\delta)} := \cup_{-\eta < t < \delta} U_t$ is open. For this purpose, define function

$$\psi(p, t, z) := \begin{bmatrix} p - t\nu(p) - z \\ -g_i(p) \end{bmatrix}, \quad \forall (p, t, z) \in \mathbb{R}^n \times (-\eta, \delta) \times \mathbb{R}^n.$$

Note that its partial Jacobian is

$$\nabla_{(p,t)}\psi(p,t,z) = \begin{bmatrix} I_n - t\nabla_p\nu(p) & -\nu(p) \\ -\nabla g_i(p)^T & 0 \end{bmatrix}.$$

From the choice of outward normal direction, we know $\nu(p) = -\frac{\nabla g_i(p)}{\|\nabla g_i(p)\|}$. So

$$\det(\nabla_{(p,t)}\psi(p,t,z)) = \|\nabla g_i(p)\| \left(\nu(p)^T \left(I_n - t\nabla_p\nu(p)\right)^{-1} \nu(p)\right) \det\left(I_n - t\nabla_p\nu(p)\right).$$

Fix an arbitrary point $p_t = p - t\nu(p) \in U_{(-\eta,\delta)}$. Then $\psi(p,t,p_t) = 0$ and $\|\nabla g_i(p)\| > 0$ (since U is smooth). If η and δ are sufficiently small, it holds that $\det(\nabla_{(p,t)}\psi(p,t,p_t)) > 0$ and hence $\nabla_{(p,t)}\psi(p,t,p_t))$ is nonsingular. By the Implicit Function Theorem, there exist a small open neighborhood $\mathcal{O}_{p_t}$ of p_t in $\mathbb{R}^n$ and a small open neighborhood $\mathcal{O}_{p,t}$ of (p,t) in $\mathbb{R}^n \times (-\eta,\delta)$ such that $\psi(w,s,q) = 0$ defines a smooth function $(w,s) = \zeta(q)$ with domain $\mathcal{O}_{p_t}$ and range $\mathcal{O}_{p,t}$. That is, for every $q \in \mathcal{O}_{p_t}$, we can find a unique (w,s) in $\mathcal{O}_{p,t}$ such that $q = w - s\nu(w)$ and $g_i(w) = 0$. If we choose the open neighborhoods $\mathcal{O}_{p_t}$ and $\mathcal{O}_{p,t}$ sufficiently small, w must be sufficiently close to p enough so that $w \in U$ and $s \in (-\eta,\delta)$. So $q \in U_{(-\eta,\delta)}$. This says $U_{(-\eta,\delta)}$ is an open set in $\mathbb{R}^n$.

Now we show that W_δ also lies to the inside of the tangent planes of U_r for all $r > 0$ small enough, by generalizing the argument in the proof in Proposition 3.3 in [5]. From the openness of $\cup_{-\eta<t<\delta}U_t$ and compactness of S, we know W_δ is compact from (5.2). For this purpose, define function $f_r : \overline{U}_0 \times W_\delta \to \mathbb{R}$ as

$$f_r(p,a) = \langle a - p_r, \nu(p_r)\rangle, \quad \forall (p,a) \in \overline{U}_0 \times W_\delta,$$

which is the signed distance between a and $T_{p_r}(\overline{U}_0)$ (See [5]). By item (6) of Claim A, for every point $p \in \overline{U}_0 = \overline{U}$, the convex set S lies to the inside of the tangent plane $T_p(\overline{U}_0)$ and $S \cap \left(T_p(\overline{U}_0)\backslash\{p\}\right) = \emptyset$. Since $W_\delta \subset S$ and $W_\delta \cap \overline{U}_0 = \emptyset$, we know W_δ lies strictly to the inside of the tangent plane $T_p(\overline{U}_0)$, meaning that it does not touch $T_p(\overline{U}_0)$. Thus $f_0 < 0$ on the compact set $\overline{U}_0 \times W_\delta$. By continuity, we know $f_r < 0$ on $\overline{U}_0 \times W_\delta$ for $r > 0$ small enough. This means the set W_δ lies strictly to the inside of all the tangent planes of $\overline{U}_r$ for $0 \leq r \leq \delta$ is sufficiently small.

Step 3 For $r \in [0,\delta]$ sufficiently small, one has

$$S \smallsetminus (\cup_{0\leq t<r}U_t) \subset W_\delta \cup (\cup_{r\leq t<\delta}U_t).$$

From Step 1, we know $\cup_{r\leq t<\delta}U_t$ lies to the inside of all the tangent planes of $\overline{U}_r$. From Step 2, we know W_δ lies to the inside of all the tangent planes of $\overline{U}_r$. So we immediately conclude that $S \smallsetminus (\cup_{0\leq t<r}U_t)$ lies to the inside of all the tangent planes of $\overline{U}_r$. $\qquad\square$

For $r > 0$ small enough, define two new sets

$$W = \mathrm{conv}\left(\overline{U}_r \cup \overline{S \smallsetminus (\cup_{0\leq t<r}U_t)}\right), \quad K = W + \bar{B}(0,r).$$

Claim D For $r > 0$ small enough, the set K is compact convex and

$$\partial K \cap \partial S = Z_i \cap \partial S.$$

Proof. Convexity and compactness are obvious. Note that

(5.3) $$\partial K = \{b : dist(b,W) = r\}.$$

First, we prove the inclusion $Z_i \cap \partial S \subset \partial K \cap \partial S$. Suppose $p \in Z_i \cap \partial S \subset \overline{U}$, then

$$dist(p, \overline{U}_r) \geq dist(p, W),$$

because $\overline{U}_r \subset W$. From item (ii) of Claim B we know the set $\overline{U}_r$ lies to the inside of the tangent plane $T_{p_r}(\overline{U}_r)$, and from Claim C we know $S \smallsetminus (\cup_{0\leq t<r}U_t)$ lies to the inside of $T_{p_r}(\overline{U}_r)$. Thus, by the definition of W, the set W also lies to the inside of $T_{p_r}(\overline{U}_r)$. Since p lies to the outside of $T_{p_r}(\overline{U}_r)$, we have

$$dist(p, T_{p_r}(\overline{U}_r)) \leq dist(p, W).$$

Then from item (iv) of Claim B we can see that

$$r = dist(p, \overline{U}_r) \ = dist(p, T_{p_r}(\overline{U}_r)) = \ dist(p, W).$$

So $dist(p, W) = r$ and hence $p \in \partial K \cap \partial S$ from (5.3). Hence it holds $Z_i \cap \partial S \subset \partial K \cap \partial S$.

Second, we prove the reverse inclusion $\partial K \cap \partial S \subset Z_i \cap \partial S$. Start by noting that

$$\partial S = (Z_i \cap \partial S) \cup \big(\partial S \setminus (Z_i \cap \partial S)\big).$$

We set about to prove $\partial S \setminus (Z_i \cap \partial S)$ lies in the interior of K. Consider $a \in \partial S \setminus (Z_i \cap \partial S)$. If $a \in S \setminus (\cup_{0 \leq t < r} U_t)$, then $a \in W$ and hence $a + B(0, r/2) \subset K$ which implies a is in the interior of K. If $a \notin S \setminus (\cup_{0 \leq t < r} U_t)$, then $a \in U_s$ for some $s \in (0, r)$ because $a \notin U_0$. By definition of U_s and U_r, there exists $b \in U_r$ such that $a = b + (r - s)\nu(q)$ for some $q \in U_0$. Since $b \in W$ and $\|a - b\| = r - s$, we know $a + B(0, s/2) \subset b + B(0, r - s/2) \subset K$ and hence a is also in the interior of K. Combining the above, we know $\partial S \setminus (Z_i \cap \partial S)$ lies in the interior of K and hence does not intersect ∂K. Thus $\partial K \cap \partial S = \partial K \cap (Z_i \cap \partial S) \subset Z_i \cap \partial S$, which completes the proof. $\square$

The proof from here on is essentially the same as in Proposition 3.3 [5], so we could refer to that but include here a slightly annotated version for convenience. The next step is to define a set K^ϵ which is a small perturbation of K and which we shall prove has the properties our lemma requires. Let $V \subset U$ be an open set with $Z_i \cap \partial S \subset V \subset U$. Set $U' = \nu(U)$, and $V' = \nu(V)$. Then U' and V' are open in $\mathbb{S}^{n-1}$, because (since the second fundamental form of U is nondegenerate) ν is a local diffeomorphism. Let $\bar\phi : \mathbb{S}^{n-1} \to \mathbb{R}$ be a smooth function with support $\text{supp}(\bar\phi) \subset U'$, and $\bar\phi|_{\overline{V}'} \equiv 1$. Let ϕ be the extension of $\bar\phi$ to $\mathbb{R}^n$ given by $\phi(0) = 0$ and $\phi(p) := \bar\phi(p/\|p\|)$, when $p \neq 0$. Define $\bar h : \mathbb{R}^n \to \mathbb{R}$ by

$$\bar h^\epsilon(p) := \tilde h^\epsilon(p) + \phi(p)(h(p) - \tilde h^\epsilon(p)),$$

where h is the support function of K, that is,

$$h(p) := \sup_{x \in K} \langle p, x \rangle$$

and $\tilde h^\epsilon$ is the *Schneider transform* of h

$$\tilde h^\epsilon(p) := \int_{\mathbb{R}^n} h(p + \|p\|x)\theta_\epsilon(\|x\|)dx.$$

Note that $\tilde h^\epsilon$ is a convex function (see Ghomi [5]). Here $\theta_\epsilon : [0, \infty) \to [0, \infty)$ is a smooth function with $\text{supp}(\theta_\epsilon) \subset [\epsilon/2, \epsilon]$ and $\int_{\mathbb{R}^n} \theta_\epsilon(\|x\|)dx = 1$. $\bar h^\epsilon$ supports the convex set

$$K^\epsilon := \{x \in \mathbb{R}^n : \langle x, p \rangle \leq \bar h^\epsilon(p), \ \forall p \in \mathbb{R}^n\}.$$

Claim E The set $T = K^\epsilon$ satisfies the conclusions of Lemma 5.4 when $\epsilon > 0$ is sufficiently small.

Proof. (i) We show K^ϵ is a convex body with support function $\bar h^\epsilon$. To see this, it suffices to check that $\bar h^\epsilon$ is positively homogeneous and convex. By definition, $\bar h^\epsilon$ is obviously homogeneous. Thus to see convexity, it suffices to show that $\nabla^2 \bar h^\epsilon(p)$ is nonnegative semidefinite for all $p \in \mathbb{S}^{n-1}$. Since $\bar h^\epsilon|_{\mathbb{S}^{n-1} \setminus U'} = \tilde h^\epsilon$, and $\tilde h^\epsilon$ is convex, we need to check this only for $p \in U'$. To this end, note that, for each $p \in U'$, $\nabla^2(h|_{T_p \mathbb{S}^{n-1}}) \succ 0$. Here T_p denotes the tangent plane at p. Further, by construction,

$$\|h - \bar h^\epsilon\|_{C^2(\overline{U}')} \to 0.$$

So, for every $p \in \overline{U}'$, there exists an $\epsilon(p) > 0$ such that $\bar h^\epsilon|_{T_p \mathbb{S}^{n-1}}$ has strictly positive Hessian. Since $\overline{U}'$ is compact and $\epsilon(p)$ depends on the size of the eigenvalues of the Hessian matrix of $\bar h^\epsilon|_{T_p \mathbb{S}^{n-1}}$, which in turn depend continuously on p, it follows that there is an $\epsilon > 0$ such that $\nabla^2(\bar h^\epsilon|_{T_p \mathbb{S}^{n-1}}) \succ 0$ for all $p \in \overline{U}'$.

(ii) We show that ∂K^ϵ is nonsingular (hence smooth) and positively curved. By Lemma 3.1 in Ghomi [5], we only need check $\nabla^2(\bar h^\epsilon|_{T_p \mathbb{S}^{n-1}}) \succ 0$ for all $p \in \mathbb{S}^{n-1}$. For $p \in U'$, this was verified above. For $p \in \mathbb{S}^{n-1} \setminus U'$, note that $\bar h^\epsilon = \tilde h^\epsilon$ on the cone spanned by $\mathbb{S}^{n-1} \setminus U'$. So it is enough to check that

$\nabla^2(\tilde{h}^\epsilon|_{T_p\mathbb{S}^{n-1}}) \succ 0$. By Lemmas 3.2 and 3.1 of Ghomi [5], this follows from the boundedness of the radii of curvature from below.

(iii) Obviously K^ϵ is compact. We show that $Z_i \cap \partial S \subset \partial K^\epsilon$. The proof is almost the same as the one of Proposition 3.3 in [5]. Since $Z_i \cap \partial S \subset U$, which is smooth in ∂K, we have $h(p) = \langle \nu^{-1}(p), p \rangle$, for all $p \in U'$. Apply the fact $\nabla h(p) = \nu^{-1}(p)$ to get

$$\nu^{-1}(p) = \nabla h(p) = \nabla \bar{h}^\epsilon(p) = \bar{\nu}^{-1}(p)$$

for all $p \in V'$, where $\bar{\nu}$ is the Gauss map of ∂K^ϵ (see the proof of Proposition 3.3 in [5]). So $Z_i \cap \partial S \subset \bar{\nu}^{-1}(V') \subset \partial K^\epsilon$.

(iv) We show that $S \cap \partial K^\epsilon = \partial S \cap \partial K^\epsilon = Z_i \cap \partial S$. Let $A := \bar{\nu}^{-1}(V')$. Then $A \subset \partial K^\epsilon$, as shown in (iii) above. Since the Gauss map is continuous, A is a relatively open subset of V. Obviously $A \subset U \subset \partial K$. So the sets $\partial K^\epsilon \backslash A$, $\partial K \backslash A$ are all compact. The set $S \backslash (\partial S \cap Z_i)$ is contained in the interior of K (this has been proved in the proof of Claim D), so $S \cap \partial K = (\partial S \cap Z_i) \cap \partial K$. From (iii) above, we know $Z_i \cap \partial S \subset A$ and hence $S \cap (\partial K \backslash A) = \emptyset$. So it holds

$$(5.4) \qquad \partial S \cap Z_i \subset A \subset \partial K^\epsilon.$$

Since $K^\epsilon \to K$ as $\epsilon \to 0$, it must hold that $\partial K^\epsilon \backslash A \to \partial K \backslash A$ as $\epsilon \to 0$. Thus, for $\epsilon > 0$ small enough, we have $S \cap (\partial K^\epsilon \backslash A) = \emptyset$, which implies (by using (5.4))

$$S \cap \partial K^\epsilon = (S \cap (\partial K^\epsilon \backslash A)) \cup (S \cap A) = S \cap A.$$

Then we can see

$$\partial S \cap Z_i \subset \partial S \cap \partial K^\epsilon \subset S \cap \partial K^\epsilon = S \cap A \subset S \cap U = \partial S \cap Z_i,$$

where the last equality is by item (4) of Claim A. So all the intersections above are the same and hence we get $S \cap \partial K^\epsilon = \partial S \cap \partial K^\epsilon = Z_i \cap \partial S$.

(v) We show that $S \subset K^\epsilon$. Let A be the relatively open subset of V defined above. Fix an interior point $v \in W \subset S$. We proceed by contradiction. If $S \not\subset K^\epsilon$, then the interior of S is not contained in the interior of K^ϵ since they are both compact. So we can find an interior point $u \in S$ but $u \notin K^\epsilon$. Since S and K^ϵ are convex, the line segment L connecting u and v must be contained in S and intersect ∂K^ϵ, say, $b \in L \cap \partial K^\epsilon$. Since u, v are both in the interior of S, b must also be an interior point of S. Thus $b \notin \partial S \cap Z_i$. We also must have $b \notin A$, because $S \cap A = \partial S \cap Z_i$. So $b \in \partial K^\epsilon \backslash A$. Since $b \in L \subset S$, we get $b \in S \cap (\partial K^\epsilon \backslash A)$, which is a contradiction since $S \cap (\partial K^\epsilon \backslash A) = \emptyset$, as shown in (iv) above. Therefore S must be contained in K^ϵ for $\epsilon > 0$ sufficiently small. $\qquad \square$

Now that Claim E is proved, the proof of Lemma 5.4 is finished. $\qquad \square$

5.3. Proof of Theorem 5.1

Given $u \in \partial S$, pick a g_i for which $g_i(u) = 0$. By assumption, if g_i is not sos-concave, then each Z_i has positive curvature at all u in $Z_i \cap \partial S$ and $\nabla g_i(u) \neq 0$. By Lemma 5.4, S_i is extendable poscurve-convex with respect to S. Apply Theorem 5.3, noting that they produce the desired Lasserre-Parrilo type moment construction, to finish the proof. $\qquad \square$

6. Conclusions

For compact convex semialgebraic sets, this paper proves the sufficient condition for semidefinite representability: each component of the boundary is nonsingular and has positive curvature, and the necessary condition: the boundary components have nonnegative curvature when nonsingular. We can see that the only gaps between them are the boundary has singular points or has zero curvature somewhere. Compactness is required in the proof for the sufficient condition, but it is not clear whether the compactness is necessary in the general case. So far, there is no evidence that SDP representable sets require more than being convex and semialgebraic. In fact, we conjecture that

Every convex semialgebraic set in $\mathbb{R}^n$ is semidefinite representable.

The results of this paper are mostly on the theoretical existence of semidefintie representations. One important and interesting future work is to find concrete conditions guaranteeing efficient and practical constructions of lifted LMIs for convex semialgebraic sets and convex hulls of nonconvex semialgebraic sets.

References

[1] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications.* MPS-SIAM Series on Optimization, SIAM, Philadelphia, 2001

[2] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

[3] J. Bochnak, M. Coste and M-F. Roy. *Real Algebraic Geometry*, Springer, 1998.

[4] D. Cox, J. Little and D. O'Shea. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra.* Third Edition. Undergraduate Texts in Mathematics, Springer, New York, 2007.

[5] M. Ghomi. Optimal Smoothing for Convex Polytopes, *Bull. London Math. Soc.* 36 (2004) 483-492 2004

[6] W. Helton and J. Nie. Semidefinite representation of convex sets. To appear in *Mathematical Programming.*

[7] W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.* **60** (2007), No. 5, pp. 654-674.

[8] J. Lasserre. Convex sets with lifted semidefinite representation. To appear in *Mathematical Programming.*

[9] Y. Nesterov and A. Nemirovskii. Interior-point polynomial algorithms in convex programming. SIAM Studies in Applied Mathematics, 13. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[10] A. Nemirovskii. Advances in convex optimization: conic programming. *Plenary Lecture, International Congress of Mathematicians (ICM)*, Madrid, Spain, 2006.

[11] P. Parrilo. Exact semidefinite representation for genus zero curves. Talk at the Banff workshop "Positive Polynomials and Optimizatio", Banff, Canada, October 8-12, 2006.

[12] R.T. Rockafellar. *Convex analysis.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997.

[13] M. Spivak. *A comprehensive introduction to differential geometry.* Vol. II, second edition, *Publish or Perish, Inc., Wilmington, Del.*, 1979.

[14] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of semidefinite programming.* Kluwer's Publisher, 2000.

Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093
E-mail address: helton@math.ucsd.edu

Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, CA 92093
E-mail address: njw@math.ucsd.edu